University of Warsaw
Faculty of Economic Sciences

Working Papers
No. 3/2013 (88)

Michał Brzeziński

Variance estimation for richness measures

Warsaw 2013

UNIVERSITY OF WARSAW
Faculty of Economic Sciences

Variance estimation for richness measures

MICHAŁ BRZEZIŃSKI
Faculty of Economic Sciences
University of Warsaw
e-mail: mbrzezinski@wne.uw.edu.pl

Abstract

Richness indices are distributional statistics used to measure the incomes, earnings, or wealth of the rich. This paper uses a linearization method to derive the sampling variances for recently introduced distributionally-sensitive richness measures when estimated from survey data. The results are derived for two cases: (1) when the richness line is known, and (2) when it has to be estimated from the sample. The proposed approach enables easy consideration of the effects of a complex sampling design. Monte Carlo results suggest that the proposed approach allows for reliable inference in case of "concave" richness indices, but that it is not satisfactory in case of "convex" richness measures.

Keywords:
richness, affluence, distributional indices, variance estimation, statistical inference

JEL:
C12, C46, D31

1. Introduction

In recent years, interest in the distributional analysis of the top part of income and wealth distributions has grown. The measures most commonly applied in literature are the richness headcount ratio (Medeiros 2006), which is the proportion of individuals in the population above the richness line and top quantile shares (e.g., Atkinson *et al*. 2011 for the case of top income shares, and Jäntti *et al*. 2008 for the case of top wealth shares). Recently, Peichl *et al*. (2010) introduced a new class of distributionally-sensitive richness indices analogous to the well-known family of additively separable poverty indices (Atkinson 1987). This class of indices is sensitive to both richness intensity and inequality among the rich.

This paper uses the variance linearization approach of Deville (1999), which is based on a modified concept of the influence function introduced by Hampel (1974), to derive the sampling variances for the richness headcount radio and distributionally-sensitive richness measures of Peichl *et al*. (2010) when estimated from survey data. The approach enables the estimation of variance of richness measures in two cases: (1) when the richness line is fixed and known, and (2) when it is defined as a multiple of the mean, median, or other quantile of the underlying distribution and estimated from the sample. Moreover, the approach allows easy consideration of the effects of any complex sampling design found in real-world surveys.

Although most recent studies on top quantile shares use income tax data, many other works use household survey data. Survey data are used to measure the top of the distribution when tax return data are unavailable or only available for a limited time (e.g., Leigh and Van der Eng 2009, Piketty and Qian 2009, Brzezinski 2010), and when one wants to compare results from tax returns and survey data (Burkhauser *et al*. 2012) or to compute measures requiring micro-data similar to those introduced by Peichl *et al*. (2010) as well as Peichl and Pestel (2011). Household survey data used in such studies often come from samples of moderate or small size. In this context, the question of sampling variability of richness statistics naturally arises.

Previous papers dealing with statistical inferences on richness indices (e.g., Brzezinski 2010, Peichl and Pestel 2011) rely on bootstrap methods. However, the wide applicability of bootstrapping comes at a price. First, bootstrapping is time consuming. For example, calculating the variances for a portfolio of nine richness indices using asymptotic methods introduced in this paper took about 3 seconds on a modern laptop, but as much as about 55 minutes by bootstrapping with 1000 replications.[1] Second, bootstrapping for complex sample surveys, with such features as probability weighting, stratification, and clustering, is usually difficult to implement in practice because it requires using non-standard methods such as rescaling or weighting (Antal and Tillé 2011). Wolter (2007, p. 365) suggests that bootstrap methods for complex surveys should be adequately tested before they can be recommended in an unqualified way. The methodology for estimating asymptotic sampling variances proposed in this paper does not suffer from these two limitations. It is fast and can be used in a standard way under any complex survey design applied in real-world surveys.[2]

This paper is organized as follows. Section 2 presents the existing measures of richness. Section 3 provides an overview of the linearization approach to variance estimation and its application to richness indices. Section 4 provides Monte Carlo evidence on the performance of the proposed method. Section 5 concludes the paper.

2. Richness indices

As noticed by Peich *et al*. (2010), using the richness headcount ratio or top quantile shares as measures of distributional affluence is associated with serious limitations.[3] Assuming that no

[1] The sample of roughly 37 000 income observations was obtained from the 2010 Polish Household Budget Survey.
[2] A Stata program estimating richness indices and their variances is available in the author's webpage.
[3] Top quantile shares may of course be used not only as measures of richness but also as distributional measures

mobility exists between rich and non-rich individuals, the former measure is insensitive to the changes in both the affluence of the rich (richness intensity) and inequality among the rich. The latter measures assume that the proportion of the rich is fixed and do not take into account the distribution of well-being among the rich. To overcome these deficiencies, Peich *et al*. (2010) introduce a new class of distributionally-sensitive richness indices analogous to the well-known family of additively separable poverty indices (Atkinson 1987). This family is sensitive to both richness intensity and inequality among the rich. Peichl *et al*. (2010) use their indices to provide a detailed analysis of income richness in Germany and a comparison of income richness among 26 European countries. Peichl and Pestel (2011) compare income and wealth affluence in Germany and the United States in 2007, as well as analyze the changes in the USA richness from 1989 to 2007. Their paper also generalizes the framework of Peich *et al*. (2010) to a multidimensional one. In addition, the measures of Peich *et al*. (2010) have been recently applied to analyze changes in income richness over time in Poland (Brzezinski 2010), and to study wealth affluence in Italy (Eisenhauer 2011).

In the remainder of this section, we formally present the richness indices of Peich *et al*. (2010). Let U be the population of N individuals with incomes denoted by $x_1,\dots,x_k,\dots,x_N$. For a given richness line ρ, an individual is rich if her income is above ρ. The richness line may be fixed and known or defined in relation to the underlying distribution and estimated from the sample.[4] The most simple richness index is the richness headcount ratio defined as a proportion of the population above the richness line and given by

$$R^{HC} = \frac{1}{N}\sum_{k\in U} \delta(x_k > \rho) = 1 - F(\rho), \qquad (1)$$

where $\delta(\cdot)$ is an indicator function equal to 1 when its argument is true and 0 otherwise, and

per se or as approximate inequality indices.

[4] See Medeiros (2006) and Eisenhauer (2011) for attempts to justify richness lines defined in relation to the poverty line. Peichl *et al*. (2010) use relative richness lines equal to twice the median income, whereas Brzezinski (2010) employs three richness lines equal to two, three, and four times the median income. Peichl and Pestel (2011) take the 80th percentile as the richness threshold.

$F(\cdot)$ is the cumulative distribution function of x_k. The richness indices introduced by Peichl *et al*. (2010) belong to the class of decomposable (additively separable) measures defined as follows

$$R = \frac{1}{N} \sum_{k \in U} h(x_k, \rho), \tag{2}$$

where $h(x_k, \rho)$ is the individual "richness function," which is 0 if $x_k \leq \rho$ and is continuous and strictly increasing when $x_k > \rho$. This class of measures satisfies several axioms (i.e., focus, continuity, monotonicity, and subgroup decomposability) taken from the axiomatic literature on poverty (e.g., Zheng 1997, Chakravarty 2009) and properly redefined for the purpose of measuring richness instead of poverty.

Peichl *et al*. (2010) notice that the familiar (minimal) transfer axiom, which requires poverty to increase after a transfer of income from a poor person to another poor person with higher income, can be translated to a richness measurement framework in two normatively justifiable ways. Accordingly, they propose a concave (convex) transfer axiom T1 (T2), according to which a richness index decreases (increases) after a rank-preserving transfer from a rich person to another rich person with higher income.[5] Their preferred "concave" class of richness measures, which satisfies the concave transfer axiom, is analogous to the family of poverty indices introduced by Chakravarty (1983) and defined as

$$R_{\beta}^{Cha} = \frac{1}{N} \sum_{k \in U} \left[1 - \left(\frac{\rho}{x_k} \right)^{\beta} \right] \delta(x_k > \rho), \beta > 0, \tag{3}$$

where β is a parameter describing a preference for richness.

The "convex" richness indices are defined by Peichl *et al*. (2010) in analogy to the popular Foster–Greer–Thorbecke (1984) (FGT) family of poverty indices satisfying axiom T2 as follows

[5] To satisfy T1 (T2), the individual richness function $h(x_k, \rho)$ in Eq. (2) has to be strictly concave (convex).

$$R_{\alpha}^{FGT,T2} = \frac{1}{N}\sum_{k\in U}\left(\frac{x_k}{\rho}-1\right)^{\alpha}\delta(x_k > \rho), \alpha > 1, \quad (4)$$

where α again expresses a degree of preference for richness (see Peichl *et al*. (2010) for a detailed comparison of "concave" versus "convex" richness measures).

Given a random household sample (of size *N*) denoted by *S*, which is a subset of *U*, and a set of survey weights w_k $(k \in S)$, the measures given in Eq. (2) can be estimated by

$$\hat{R} = \frac{1}{\sum_{k\in S} w_k}\sum_{k\in S} w_k h(x_k, \rho), \quad (5)$$

if ρ is known, or by substituting an estimator $\hat{\rho}$ for ρ otherwise.

3. Variance estimation by linearization

3.1. Overview of the approach

When richness line is known, sampling variances of indices defined in Eq. (2) can be estimated in a straightforward manner that is analogous to the derivation of sampling variances of decomposable poverty indices (Kakwani 1993, Bishop *et al*. 1995). However, if the richness line is estimated from the sample, one needs to take into account additional sampling variability of the richness line; therefore, other methods are needed.[6] Existing variance estimation approaches can be broadly divided into resampling techniques and linearization methods (Wolter 2007). In this paper, we use a linearization method of Deville (1999) that is easy to use and provides a powerful variance-estimation tool for complex statistics applied to any sampling design. Recently, Deville's method has been applied by Osier (2009), Langel and Tille (2011), as well as Verma and Betti (2011) to derive sampling variances of various poverty and inequality measures. The approach is based on a slightly modified concept of the influence function introduced in the field of robust statistics by Hampel (1974); see also Hampel *et al*. (1986).

[6] In particular, see Zheng (2001), who derives sampling variances for decomposable poverty indices with relative poverty lines.

According to Deville's approach, a population parameter of interest, θ, can be written as a functional $T(M)$, where M is a finite and discrete measure that allocates a unit mass to all $k \in U$. For example, the total Y of a variable y can be expressed as $Y = \sum_{k \epsilon U} y_i M(i) = \int y dM = T(M)$. The influence function of the functional $T(M)$, $I[T(M)]$, is defined by Deville as the following derivative

$$z_k = I[T(M)]_k = \lim_{\varepsilon \to 0} \frac{T(M + \varepsilon\delta_k) - T(M)}{\varepsilon}, \tag{6}$$

where δ_k is the unit mass for unit k. The influence function measures the influence of unit *k* on θ by adding an infinitesimal contamination, ε, to a given observation. Under the asymptotic assumptions provided by Deville (1999), the variance of the estimator of θ, $\hat{\theta}$, can be approximated by the variance of the total of z_k given by

$$\text{var}(\hat{\theta}) \approx \text{var}\left(\sum_{k \in S} z_k w_k\right). \tag{7}$$

The pseudovariable z is then called a *linearized variable*, and the sampling variance of a total $\sum_{k \in S} z_k w_k$ may be estimated by standard survey sampling methods, thereby enabling the consideration of any actual sampling design used (e.g., Cochran 1977, Deaton 1997). Deville (1999) also provides several derivation rules that follow the rules of differential calculus and simplify the task of obtaining linearized variables such that tedious limit calculations in Eq. (6) can be avoided.

3.2. Linearization for richness indices

If the richness line ρ need not be estimated from the sample, linearized variables for the variance estimation of richness indices can be obtained by applying Deville's (1999, p. 197) derivation rule for the linearization of a ratio to Eq. (2) and written as

$$z_k = I(R|\rho = \text{const.})_k = \frac{1}{N}[h(x_k, \rho) - R]. \quad (8)$$

However, if a richness line is defined in relation to a quantile or the mean of the underlying distribution, then further linearization is required. Further linearization can be performed using Deville's (1999, p. 198) *rule 7* for the linearization of a functional with a parameter. The application of this rule yields the linearized variable defined by

$$z_k = I(R)_k = I(R|\rho = \text{const.})_k + \frac{\partial R}{\partial \rho} I(\rho)_k. \quad (9)$$

Equations analogous to the Eqs. (8) and (9) have been previously derived by Cowell and Victoria-Feser (1996) in the context of analysing the robustness of additively separable poverty indices. The first term of Eq. (9) is equivalent to Eq. (8) and gives the influence function of the richness index R assuming that the richness line is fixed, whereas the second term accounts for the influence of the richness line. The latter consists of two components, namely, a partial derivative of a richness index with respect to the richness line, and the influence function of the richness line. Table 1 reports the formulae for the partial derivatives of our richness indices with respect to the richness line (see Appendix A for the derivation).

Table 1. Linearization component $\partial R/\partial \rho$ for richness indices

Richness index	$\partial R/\partial \rho$
$R^{HC} = \frac{1}{N}\sum_{k\in U} \delta(x_i > \rho) = 1 - F(\rho)$	$-f(\rho)$
$R^{Cha}_{\beta} = \frac{1}{N}\sum_{k\in U} \left[1 - \left(\frac{\rho}{x_k}\right)^{\beta}\right] \delta(x_k > \rho), \beta > 0$	$\frac{1}{N}\sum_{k\in U} -\frac{\beta}{\rho}\left(\frac{\rho}{x_k}\right)^{\beta} \delta(x_k > \rho)$
$R^{FGT,T2}_{\alpha} = \frac{1}{N}\sum_{k\in U} \left(\frac{x_k}{\rho} - 1\right)^{\alpha} \delta(x_k > \rho), \alpha > 1$	$\frac{1}{N}\sum_{k\in U} -\frac{\alpha x_k}{\rho(x_k - \rho)}\left(\frac{x_k}{\rho} - 1\right)^{\alpha} \delta(x_k > \rho)$

Using Eq. (9) for variance linearization of the richness headcount involves the derivative of the cumulative distribution function F. Given that F is a discontinuous step function, Deville

(1999) suggests that it should be replaced by its smoothed version $\widetilde{F}$. Then, the derivative of $\widetilde{F}$ at x_k is the density function corresponding to $F, f(x_k)$. Following Berger and Skinner (2003) as well as Osier (2009), we use Gaussian kernel smoothing to estimate $f(x_k)$.

Linearization of the richness line estimated from the sample depends on how this line is defined. For the richness line defined as $\rho = \gamma\xi_q$, where ξ_q is a quantile of order q (i.e., $\xi_q = \sup\{x_k | F(x_k) \le q\}$) and $\gamma ... \ge 1$, the linearized variable is given by Deville (1999) as

$$z_k = I(\gamma\xi_q)_k = \gamma I(\xi_q)_k = \frac{-\gamma[\delta(x_k < \xi_q) - q]}{f(\xi_q)N}. \tag{10}$$

On the other hand, if the richness line is defined as a multiple of the mean income (μ), the influence function can be written as

$$z_k = I(\gamma\mu)_k = \gamma I(\mu)_k = \frac{\gamma}{N}(x_k - \mu). \tag{11}$$

A linearized variable specific to a given choice of richness index R and type of richness line can be obtained by substituting the appropriate partial derivative from Table 1 and either Eq. (10) or Eq. (11) into Eq. (9). For example, the linearized variable for the $R_{0.5}^{Cha}$ index and richness line equal to thrice the median income, $\rho = 3\xi_{0.5}$, can be written as follows

$$\begin{aligned} z_k = \frac{1}{N}\left(\left[1 - \left(\frac{3\xi_{0.5}}{x_k}\right)^{0.5}\right]\delta(x_k > 3\xi_{0.5}) - R_{0.5}^{Cha}\right) \\ + \frac{1}{N}\left(\frac{-3[\delta(x_k < \xi_{0.5}) - 0.5]}{f(\xi_{0.5})}\sum_{k\in U} - \frac{0.5}{3\xi_{0.5}}\left(\frac{3\xi_{0.5}}{x_k}\right)^{0.5}\delta(x_k > 3\xi_{0.5})\right). \end{aligned} \tag{12}$$

Finally, a linearization variance estimator can be determined by replacing $N, \rho, R, f(\cdot)$ and either ξ_q or μ by $\widehat{N} = \sum_{k\in S} w_k, \widehat{\rho}, \widehat{R}, \widehat{f}(\cdot)$ and either $\widehat{\xi}_q$ or $\widehat{\mu} = 1/\widehat{N}\sum_{k\in S} w_k x_k$.

4. Simulation results

The linearization approach to variance estimation, like other asymptotic approaches, relies on the assumption that the sample size is large enough. However, it has been shown by Davidson and Flachaire (2007) that asymptotic inference for some popular distributive indices, especially for some popular inequality measures such as the Theil index, is not accurate even in relatively large samples. They have argued that the reason of the poor performance of asymptotic inference was its extreme sensitivity to the details of the upper tail of a distribution.[7] Taking into account the fact that the richness measures are constructed using only information from the upper part of a distribution, the problems observed by Davidson and Flachaire (2007) in the context of inequality measures may be equally important or even intensified in the case of richness indices.

In this section, we study finite-sample performance of asymptotic inference for richness measures. Using estimate of a given richness index, $\widehat{R}$, given by Eq. (5) and estimate of its variance, $\widehat{V}(\widehat{R})$, obtained using Eq. (8), we can construct the following asymptotic *t*-type statistics for a hypothesis that, for some given value R_0, $R = R_0$

$$W = (\hat{R} - R_0)/[\hat{V}(\hat{R})]^{0.5}. \tag{13}$$

In our simulations, we use data drawn from the Pareto model, which is traditionally used in modelling upper tails of income and wealth distributions (Kleiber and Kotz, 2003). The c.d.f. of the Pareto model is

$$F(x) = 1 - (x/x_0)^{-\theta}, \qquad x \geq x_0 > 0, \tag{14}$$

where θ is a shape parameter and x_0 is a scale parameter. The right tail is heavier as θ decreases. We use $\theta = 1.5, 2, 2.5$, $x_0 = 1$ and $\rho = 3\xi_{0.5}$ in our simulations.[8] The sample sizes are $n = 100$,

[7] They have also found that the standard bootstrap inference for the Theil index is also poor. On the other hand, asymptotic and standard bootstrap inference for poverty measures is satisfactory.

[8] Simulations for other richness lines, namely for $\rho = \gamma\xi_{0.5}$, and $\gamma = \{2, 4\}$, produced similar results.

200, 500, 1 000, 2 000, 3 000, 4 000, 5 000, and 10 000. It can be shown that the richness indices for the distribution with the c.d.f. defined in Eq. (14) are

$$\begin{aligned} &R^{HC} = (x_0/\rho)^{\theta}, \\ &R_{\beta}^{Cha} = (x_0/\rho)^{\theta}\beta/(\theta+\beta), \\ &R_{\alpha}^{FGT,T2} = \theta(x_0/\rho)^{\theta}\Gamma(\theta-\alpha)\Gamma(1+\alpha)/\Gamma(1+\theta), \qquad \theta > \alpha, \end{aligned} \tag{15}$$

where Γ is the gamma function. We have calculated an asymptotic P value from Eq. (13) replacing R_0 by an appropriate value calculated using Eq. (15) and the Student distribution with n degrees of freedom.

We also compare the performance of asymptotic inference with standard bootstrap method known as the percentile-*t* or bootstrap-*t* method. The test is constructed as follows. First, we compute W statistics as given by Eq. (13) from the original sample and then we draw B bootstrap samples of the same size as the original sample. We set B to 199 in our simulations. For each bootstrap sample j, $j = 1, \ldots, 199$, we compute W_j^* statistic in the same way as W was computed from the original sample, but with R_0 replaced by the index $\widehat{R}$ estimated from the original sample. The bootstrap P value is the proportion of the bootstrap samples for which the bootstrap statistic W_j^* is more extreme than W.[9]

Table 2 shows the results of our simulations in terms of errors in rejection probability (ERPs), that is, the difference between the actual and nominal probabilities of rejection (set to 5%). The number of Monte Carlo replications is 10 000. The "convex" richness measure $R_{\alpha}^{FGT,T2}$ can be calculated for the Pareto model only when $\theta > \alpha$. It is also worth noting here, that the variance of the Pareto model with $\theta \leq 2$ does not exist. For this reason, bootstrap inference in this case may be invalid.[10]

[9] Notice that this type of bootstrapping requires computation of the asymptotic variance of richness indices for each bootstrap replication.

[10] This is particularly relevant for richness measures computed with richness line set to a multiple of the mean income. As shown by Athreya (1987), the bootstrap distribution of the sample mean does not converge to a deterministic distribution as the sample size goes to infinity. More generally, the bootstrap inference may fail for heavy-tailed distributions and extreme value statistics (Qi 2008).

Table 2. ERPs of asymptotic and bootstrap tests, nominal significance level is 5%

	R^{HC}			$R_{0.5}^{Cha}$			R_{3}^{Cha}			$R_{1.5}^{FGT,T2}$		$R_{2}^{FGT,T2}$
	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 2.5$
Asymptotic												
100	0.0259	0.0284	0.0812	0.0492	0.0935	0.1275	0.0232	0.0485	0.0684	0.5320	0.4423	0.6558
200	0.0052	0.0272	0.0734	0.0251	0.0521	0.0906	0.0112	0.0306	0.0531	0.4851	0.3941	0.6489
500	0.0003	0.0154	0.0012	0.0107	0.0227	0.0432	0.0073	0.0115	0.0212	0.4376	0.3243	0.5981
1 000	0.0014	0.0008	0.0062	0.0081	0.0133	0.0187	0.0037	0.0075	0.0111	0.4104	0.2720	0.5625
2 000	0.0034	0.0015	0.0053	0.0000	0.0045	0.0083	0.0009	0.0038	0.0059	0.3794	0.2326	0.5294
3 000	-0.0008	-0.0025	0.0036	0.0015	0.0032	0.0099	-0.0004	-0.0002	0.0038	0.3689	0.2113	0.5203
4 000	-0.0037	0.0031	0.0018	0.0046	0.0068	0.0072	-0.0007	0.0026	0.0036	0.3563	0.2034	0.5017
5 000	-0.0026	-0.0039	-0.0008	-0.0023	0.0025	0.0039	-0.0022	-0.0013	0.0028	0.3514	0.1889	0.4968
10 000	-0.0014	0.0031	-0.0018	0.0036	0.0031	0.0038	0.0029	0.0026	0.0032	0.3405	0.1640	0.4801
Standard bootstrap												
100	-0.0400	-0.0442	-0.0126	-0.0258	-0.0346	-0.0115	-0.0378	-0.0430	-0.0131	0.1399	0.0337	0.0933
200	-0.0031	-0.0106	-0.0446	-0.0114	-0.0193	-0.0288	-0.0082	-0.0327	-0.0404	0.2283	0.0919	0.2179
500	-0.0025	0.0052	-0.0125	-0.0003	-0.0031	-0.0093	0.0009	-0.0023	-0.0129	0.2598	0.1496	0.3317
1 000	-0.0012	0.0016	0.0030	0.0028	0.0005	-0.0043	0.0012	0.0018	-0.0043	0.2602	0.1484	0.3385
2 000	-0.0007	-0.0017	0.0010	-0.0030	-0.0024	-0.0026	-0.0015	0.0004	-0.0005	0.2613	0.1414	0.3412
3 000	-0.0012	-0.0004	-0.0013	-0.0021	0.0003	0.0029	-0.0004	-0.0012	0.0002	0.2585	0.1368	0.3425
4 000	-0.0014	0.0016	-0.0008	0.0026	0.0033	0.0032	-0.0009	0.0003	0.0010	0.2494	0.1324	0.3337
5 000	-0.0018	-0.0026	-0.002	-0.0024	-0.0004	-0.0001	-0.0013	-0.0010	0.0010	0.2534	0.1257	0.3363
10 000	0.0016	0.0023	0.0002	0.0025	0.0014	0.0021	0.0032	0.0009	0.0015	0.2477	0.1164	0.3333

However, following Davidson and Flachaire (2007) and Cowell and Flachaire (2007) we investigate in our simulations the finite-sample behaviour of the bootstrap even in the infinite-variance case.

The results from Table 2 suggest that the asymptotic inference for the richness headcount ratio and the "concave" richness indices is satisfactory even for samples of moderate size (i.e. 1 000 or larger). In many cases, standard bootstrap inference gives a small improvement over the asymptotic inference, but both approaches can be considered reliable for samples of 1 000 and more.

The performance of the asymptotic inference for "convex" richness measures is very poor. The ERPs in this case exceed 0.15 even in very large samples. Using standard bootstrap methods decreases EPRs somewhat, but still they remain unacceptably large.

One of the reasons for the dismal performance of asymptotic and standard bootstrap inferences may be their sensitivity to the presence of extreme income observations as diagnosed by Davidson and Flachaire (2007) in the context of inference for the Theil index of inequality. As a remedy, Davidson and Flachaire (2007) propose semi-parametric bootstrap, which combines a parametric modelling of the upper tail with a standard non-parametric bootstrap for the rest of the distribution.[11]

Table 3 presents simulation results for the semi-parametric bootstrap inference on the "convex" richness measures. The ERPs are significantly reduced in the case of the $\boldsymbol{R}_{\mathbf{1.5}}^{\boldsymbol{FGT},\boldsymbol{T2}}$ index and θ = 2, but only in very large samples. For data drawn from Pareto model with lighter tail (θ = 2.5), the test starts to underreject the null. The inference with a "convex" richness index putting even more weight (α = 2) on extreme observations is unreliable even in very large samples.

Overall, we conclude that the asymptotic inference on richness indices proposed in this

[11] In our implementation of the semi-parametric bootstrap, we follow closely Davidson and Flachaire's (2007) algorithm.

paper is satisfactory in the case of the richness headcount ratio and "concave" richness indices. For the "convex" richness measures both asymptotic and standard bootstrap methods are not accurate. The performance of the bootstrap inference can be improved in some cases using a semi-parametric bootstrap, especially when a parameter describing preference for richness is low.

Table 3. ERPs of semi-parametric bootstrap tests, nominal significance level is 5%

	$R_{1.5}^{FGT,T2}$		$R_{2}^{FGT,T2}$
	θ = 2	θ = 2.5	θ = 2.5
100	0.4695	0.4117	0.6644
200	0.4066	0.3183	0.5956
500	0.2935	0.1831	0.5018
1 000	0.2197	0.0982	0.4252
2 000	0.1507	0.0301	0.3413
3 000	0.1078	-0.0005	0.2943
4 000	0.0860	-0.0134	0.2704
5 000	0.0613	-0.0275	0.2295
10 000	0.0116	-0.0431	0.1608

5. Conclusions

This paper used a linearization approach to provide variance estimators for recently introduced distributionally-sensitive richness indices (Peichl *et al*. 2010) when estimated from survey data. The proposed methods can be used both when the richness line is known and when it has to be estimated from the sample. Moreover, our approach can be easily applied to any complex sampling design found in real-world surveys.

Our Monte Carlo simulations suggest that the proposed approach gives accurate inference in case of the richness headcount ratio and the "concave" richness indices. We have also found that our asymptotic approach, as well as the standard bootstrap inference, is not reliable for the "convex" richness measures. The performance of the bootstrap can be improved in some cases using a semi-parametric bootstrap procedure of Davidson and Flachaire (2007). However, since inference for "convex" richness measures, especially with higher values of the parameter describing a preference for richness, is unreliable or difficult to implement, we recommend that the "convex" measures should be used cautiously in empirical work on measuring richness.

A final remark concerns the nature of the upper-tail survey data from which richness indices and their variances are estimated. As is well known, top income or wealth survey data may be less reliable than data from other parts of the distribution because of the higher rate of non-response of the rich and higher under-reporting of some income or wealth items. This problem is more relevant to richness measures than to most of other distributional statistics (such as inequality indices) because richness measures are exclusively calculated based on more-or-less broadly defined upper tail data. If over-sampling of the rich is unavailable or insufficient, then one can use (semi-)parametric (robust) modeling of the upper tail to establish more appropriate methods of estimation and inference (Cowell and Flachaire 2007, Cowell and Victoria-Feser 2007).

References

Antal, E. and Tillé, Y. (2011) "A direct bootstrap method for complex sampling designs from a finite population," *Journal of the American Statistical Association*, vol. 106(494), pp. 534-543.

Athreya, K. B. (1987) „Bootstrap of the mean in the infinite variance case," *Annals of Statistics*, Vol. 15, No. 2, pp. 724–31.

Atkinson, A.B. (1987) "On the measurement of poverty," *Econometrica*, Vol. 55, No. 4, pp.

749–764.

Atkinson, A. B., T. Piketty, and E. Saez (2011) “Top Incomes in the Long Run of History,” *Journal of Economic Literature*, Vol. 49, No. 1, pp. 3–71.

Berger, Y.G. and C.J. Skinner (2003) “Variance estimation for a low income proportion,” *Applied Statistics*, Vol. 52, No. 4, pp. 457–468.

Bishop, J.A., K.V. Chow, and B. Zheng (1995) “Statistical inference and decomposable poverty measures,” *Bulletin of Economic Research*, Vol. 47, No. 4, pp. 329–340.

Brzezinski, M. (2010) “Income Affluence in Poland,” *Social Indicators Research*, Vol. 99, No. 2, pp. 285–299.

Burkhauser, R. V., Feng, S., Jenkins, S. J., Larrimore, J. (2012) „Recent Trends in Top Income Shares in the United States: Reconciling Estimates from March CPS and IRS Tax Return Data”, *Review of Economics and Statistics*, vol. 94(2), 371–388.

Chakravarty, S.R. (1983) “A new index of poverty,” *Mathematical Social Sciences*, Vol. 6, No. 3, pp. 307–313.

Chakravarty, S. R. (2009) *Inequality, Polarization and Poverty: Advances in Distributional Analysis*, New York: Springer.

Cochran, W.G. (1977) *Sampling techniques*, New York: John Wiley & Sons Inc.

Cowell, F. A. (2011) *Inequality among the wealthy*, CASE Papers case150, Centre for Analysis of Social Exclusion, LSE.

Cowell, F.A. and E. Flachaire (2007) “Income distribution and inequality measurement: the problem of extreme values,” *Journal of Econometrics*, Vol. 141, No. 2, pp. 1044–1072.

Cowell, F.A. and M.P. Victoria-Feser (1996) “Poverty measurement with contaminated data: A robust approach,” European Economic Review, Vol. 40, No. 9, pp. 1761–1771.

Cowell, F.A. and M.P. Victoria-Feser (2007) “Robust stochastic dominance: A semi-parametric approach,” *Journal of Economic Inequality*, Vol. 5, No. 1, pp. 21–37.

Davidson, R., & Flachaire, E. (2007) "Asymptotic and bootstrap inference for inequality and poverty measures," *Journal of Econometrics*, Vol. 141, No. 1, pp. 141–166.

Deaton, A. (1997) *The Analysis of Household Surveys: a Microeconometric Approach to Development Policy*, Baltimore, MD.: Johns Hopkins University Press.

Deville, J.C. (1999) "Variance estimation for complex statistics and estimators: linearization and residual techniques," *Survey Methodology*, Vol. 25, No. 2, pp. 193–204.

Eisenhauer, Joseph G. (2011) "The rich, the poor, and the middle class: Thresholds and intensity indices," *Research in Economics*, Vol. 65, No. 4, pp. 294–304.

Foster, J., J. Greer, and E. Thorbecke (1984) "A Class of Decomposable Poverty Measures," *Econometrica*, Vol. 52, No. 3, pp. 761–66.

Hampel, F. (1974) „The influence curve and its role in robust estimation," *Journal of the American Statistical Association*, Vol. 69(346), pp. 383–393.

Hampel, F.R., E.M. Ronchetti, P.J. Rousseeuw, and W.A. Stahel (1986) *Robust Statistics: The Approach Based on Influence Functions*, New York: Wiley.

Jäntti, M., E. Sierminska, and T. Smeeding (2008) "How is household wealth distributed? Evidence from the Luxembourg Wealth Study," in *Growing Unequal? Income Distribution and Poverty in OECD Countries*, Paris: OECD, pp. 253–278.

Kakwani, N. (1993) "Statistical inference in the measurement of poverty," *The Review of Economics and Statistics*, Vol. 75, No. 4, pp. 632–639.

Kleiber, C. and Kotz, S. (2003) *Statistical Size Distributions in Economics and Actuarial Sciences*, New Jersey: John Wiley & Sons.

Langel, M. and Y. Tillé (2011) "Statistical inference for the quintile share ratio," *Journal of Statistical Planning and Inference*, Vol. 141, No. 8, pp. 2976–2985.

Leigh, A. and P. Van der Eng (2009) "Inequality in Indonesia: What can we learn from top incomes?" *Journal of Public Economics*, Vol. 93, No. 1–2, pp. 209–212.

Medeiros, M. (2006) "The rich and the poor: the construction of an affluence line from the poverty line," *Social Indicators Research*, Vol. 78, No. 1, pp. 1–18.

Osier, G. (2009) "Variance estimation for complex indicators of poverty and inequality using linearization techniques," *Survey Research Methods*, Vol. 3, No. 3, pp. 167–195.

Peichl, A., T. Schaefer, and C. Scheicher (2010) "Measuring richness and poverty: A micro data application to Europe and Germany," *Review of Income and Wealth*, Vol. 56, No.3, pp. 597–619.

Peichl, Andreas and Nico Pestel (2011) "Multidimensional Affluence: Theory and Applications to Germany and the US," IZA Discussion Papers 5926.

Piketty, T. and N. Qian (2009) "Income Inequality and Progressive Income Taxation in China and India, 1986–2015," *American Economic Journal: Applied Economics*, Vol. 1, No. 2, pp. 53–63.

Qi, Y. (2008) "Bootstrap and empirical likelihood methods in extremes," *Extremes*, Vol. 11, No. 1, pp. 81–97.

Verma, V. and G. Betti (2011) "Taylor linearization sampling errors and design effects for poverty measures and other complex statistics," *Journal of Applied Statistics*, Vol. 38, No. 8, pp. 1549–1576.

Wolter, K.M. (2007) *Introduction to Variance Estimation*, New York: Springer Verlag.

Zheng, B. (1997) "Aggregate poverty measures," *Journal of Economic Surveys*, Vol. 11, No. 2, pp. 123–162.

Zheng, B. (2001) "Statistical inference for poverty measures with relative poverty lines," *Journal of Econometrics*, Vol. 101, No. 2, pp. 337–356.

Appendix A. Derivation of the linearization component $\partial R/\partial\rho$ for richness indices

The partial derivatives $\partial R/\partial\rho$ for richness indices can be derived as follows.

$$\frac{\partial R_{\beta}^{Cha}}{\partial\rho}=\frac{1}{N}\sum_{k\in U}\left(\frac{\partial(1-[\rho/x_k]^{\beta})}{\partial\rho}\right)\delta(x_k>\rho)=\frac{1}{N}\sum_{k\in U}\left(-\frac{\beta}{x_k}\left(\frac{\rho}{x_k}\right)^{\beta-1}\right)\delta(x_k>\rho)=$$
$$\frac{1}{N}\sum_{k\in U}\left(-\frac{\beta}{\rho}\left(\frac{\rho}{x_k}\right)^{\beta}\right)\delta(x_k>\rho)$$

$$\frac{\partial R_{\alpha}^{FGT,T2}}{\partial\rho}=\frac{1}{N}\sum_{k\in U}\left(\frac{\partial(x_k/\rho-1)^{\alpha}}{\partial\rho}\right)\delta(x_k>\rho)=\frac{1}{N}\sum_{k\in U}\left(-\frac{1}{\rho^2}\alpha x_k\left(\frac{x_k}{\rho}-1\right)^{\alpha-1}\right)\delta(x_k>\rho)=$$
$$\frac{1}{N}\sum_{k\in U}-\frac{\alpha x_k}{\rho(x_k-\rho)}\left(\frac{x_k}{\rho}-1\right)^{\alpha}\delta(x_k>\rho)$$

Faculty of Economic Sciences
University of Warsaw
44/50 Długa St.
00-241 Warsaw
www.wne.uw.edu.pl