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# Decomposition of Bivariate Inequality Indices by Attributes Revisited

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## Abstract

Decomposability of multidimensional inequality indices by attributes is considered a highly desired property. Naga and Geoffard (2006) provided for it in case of three bivariate indices. To this end, they introduced the notion of a copula function into inequality measurement theory which, as a measure of association, is a natural concept for the study of decomposability. We show that the decomposition obtained is unrelated to copulas, and prove that two indices do not admit decomposition if association is indeed measured via copula. Most notably, the proof reveals a necessary property of indices decomposable via copulas which is similar to well-known separability property.

**Keywords:**
multidimensional inequality; decomposition by attributes; copula function

**JEL:**
D31, D63

# 1 Definitions and notation

Decomposition of multidimensional inequality indices by attributes allows us to judge how much of the overall inequality comes from inequality in different attributes (income, years of schooling etc.). Although two observation units can have the same level of joint inequality, its sources can be very different and require thus different treatment, therefore this type of knowledge is useful in policy making and, in general, in comparisons of inequality patterns across regions, nations and points in time.

Following Naga and Geoffard (2006) (henceforth NG), a population consists of $n$ individuals and each individual $i$ has resources $x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2$. The joint distribution is represented by matrix $X := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in M^n$, the set of all $n \times 2$ matrices with strictly positive elements. We also write $X =: [X_1, X_2]$, where $X_j$ gives the distribution for the $j$-th attribute (an $n \times 1$ vector). A bidimensional inequality index is a real valued function $I : M^n \to \mathbb{R}_+$. Underlying the index is a social welfare function $W : M^n \to \mathbb{R}$ of the form: $W(X) = \frac{1}{n}\sum_{i=1}^{n} u(x_i)$, where $u$ is a real-valued utility function with standard properties. Given the distribution $X = [X_1, X_2]$ we define $\mu_1 := \mu(X_1), \mu_2 := \mu(X_2)$ to be means of $X_1$ and $X_2$ respectively. Let now $\theta := \theta(X)$ be such that $W(x) = u(\theta\mu_1, \theta\mu_2)$, that is, for distribution $X$, $\theta(X)$ is a fraction of the sum-total of each attribute that would give the same level of welfare as $X$, provided each attribute were distributed equally. The corresponding inequality index is given by $I(X) := 1 - \theta(X)$. NG consider three different forms of utility function and hence, three different indices, which will be presented later.

Denote by $F_{12}$ the cumulative distribution function (cdf) of $X$ and by $F_j$ the cdf of the marginal distribution of the $j$-th attribute $X_j$. By Sklar's theorem[1] there exists a copula function $c : [0,1] \times [0,1] \to [0,1]$ such that for any $(x_1, x_2) \in \mathbb{R}^2, F_{12}(x_1, x_2) = c[F_1(x_1), F_2(x_2)]$. Copula and marginal distributions characterize joint distribution fully and copula captures the association between the two variables, therefore it seems to be indeed a very useful mathematical concept to be used for decomposition.

We say that an index $\theta$ is *decomposable by attributes* if it can be decomposed into indices $\theta_1, \theta_2$ and a measure of association $\kappa$, that is, if there exist

[1] It can be found for instance in Klement and Mesiar, 2005: theorem 14.2.3.

four $\mathbb{R} \to \mathbb{R}$ functions $h; g^1; g^2; \chi$, where $h; g^1; g^2$ are monotonically increasing, such that:[2]

$$h[\theta(F_{12})] = g^1[\theta_1(F_1)] + g^2[\theta_2(F_2)] + \chi[\kappa(c)] \tag{1}$$

## 2 Decomposition by attributes and discrete copulas

It is known that for copulas defined on $n$-atomic simple distributions (such as the one we have here), there exists a bijection between the set of copulas and the set of $n \times n$ permutation matrices (Proposition 7.3.24, Klement and Mesiar, 2005; see also Mesiar, 2005). There are $n!$ copulas on $n$-atomic distributions. Furthermore, copulas change when the relative order between attributes changes. Proposition 1 in NG gives explicit expression for $\kappa$, which is:

$$\kappa = \frac{n \sum_i x_{i1}^{\alpha} x_{i2}^{\beta}}{\sum_i x_{i1}^{\alpha} \sum_i x_{i2}^{\beta}} \tag{2}$$

Let us now make the maximum element in attribute $X_1$ higher, $\kappa$ changes then but the relative order of two attributes does not change so copula does not change either. It is impossible then that $\kappa$ measures the association of attributes via copula function, because it necessarily uses information about the joint distribution other than that which is encoded in copula.

If we correct (1), that is, we make $\kappa$ indeed dependent on the joint distribution only through copula $c$, then two out of three indices presented in NG

[2]A careful reader will note the difference between the above definition and the one used in NG, which is:

$$h[\theta(F_{12})] = g^1[\theta_1(F_1)] + g^2[\theta_2(F_2)] + \chi[\kappa(c(F_1, F_2))]$$

Clearly, definition in NG is flawed. Since there is a unique copula (to be precise, it is unique on the image of $F_j$) associated with $F_{12}$ the only meaningful interpretation of $c(F_1, F_2)$ is that it is in fact $c_{F_{12}}(F_1, F_2)$. This however equals $F_{12}$ (as denoted above), so we can *always* perform such decomposition by simply subtracting the inequality in marginal distributions ($\theta_1, \theta_2$ or their functions) from an overall inequality ($\theta(F_{12})$ or its function); what is left potentially depends on the entire information about a given distribution. However, the reason for us to decompose indices at all is exactly because we want an overall inequality to be decomposed into components which *do not* depend on the entire information. Otherwise, every index is trivially decomposable.

are not decomposable. As we now present, one index is decomposed trivially. Let utility function be of the form: $u(a,b) = \alpha \ln a + \beta \ln b,\ \alpha, \beta > 0$. The definition of $\theta(X)$ implies

$$\alpha(\ln \theta(X) + \ln \mu_1) + \beta(\ln \theta(X) + \ln \mu_2) = W(X)$$

and hence

$$\ln \theta(X) = (\alpha + \beta)^{-1} \left( W(X) - \alpha \ln \mu_1 - \beta \ln \mu_2 \right)$$

Finally the corresponding index is

$$I(X) = 1 - \exp \left\{ (\alpha + \beta)^{-1} \left( W(X) - \alpha \ln \mu(X_1) - \beta \ln \mu(X_2) \right) \right\}$$

We can easily observe that

$$W(X) = \alpha \underbrace{\frac{1}{n} \sum_{i=1}^{n} \ln x_{i1}}_{=:V(X_1)} + \beta \underbrace{\frac{1}{n} \sum_{i=2}^{n} \ln x_{i2}}_{=:V(X_2)}$$

The index can be written as

$$I(x) = 1 - \exp \left\{ \frac{\alpha \left( V(X_1) - \ln \mu(X_1) \right) + \beta \left( V(X_2) - \ln \mu(X_2) \right)}{\alpha + \beta} \right\}$$

Let $J$ be the univariate entropy index [3]

$$J(x) = \frac{1}{n} \sum_{i=1}^{n} \ln \left( \frac{\mu(x)}{x_i} \right),$$

where $\mu(x)$ is the mean value of vector $x$. Then our decomposition takes the form

$$I(x) = 1 - \exp \left\{ - \frac{\alpha J(X_1) + \beta J(X_2)}{\alpha + \beta} \right\}.$$

As we can see, due to the additively separable utility function underlying this index there is not any association between attributes.

[3] Also known as the Theil second measure or Mean Logarithmic Deviation, see Theil (1967).

## 3 Non-decomposability of indices using copulas

We now prove that two indices presented in Naga and Geoffard (2006) are not decomposable if association between attributes is to be measured by copulas as postulated by the authors. We present the proof for one of the indices as the other one is similar in reasoning.[4] Let utility function be of the form $u(a,b) = a^\alpha b^\beta,\ \alpha,\beta > 0, \alpha+\beta < 1.$. The definition of $\theta$ implies

$$(\mu(X_1)\theta(X))^\alpha \, (\mu(X_2)\theta(X))^\beta = W(X)$$

Easily we get

$$\theta(X) = \left(\frac{W(X)}{\mu(X_1)^\alpha \mu(X_2)^\beta}\right)^{\frac{1}{\alpha+\beta}}$$

and the corresponding index is

$$I(X) = 1 - \left(\frac{W(X)}{\mu(X_1)^\alpha \mu(X_2)^\beta}\right)^{\frac{1}{\alpha+\beta}}$$

We will now show that this index does not admit any decomposition of the form

$$I(X) = k(I_1(X_1), I_2(X_2), c(X)), \tag{3}$$

where $I_1, I_2$ are indices for one attribute, $c(X)$ is the unique discrete copula corresponding to the distribution of $X$ and $k$ is a strictly coordinate-wise increasing function. To this end we will find $X_1, X_1', X_2, X_2'$ such that the copulas corresponding to distributions $[X_1, X_2], [X_1, X_2']$ and to $[X_1', X_2], [X_1', X_2']$ are the same and

$$I(([X_1, X_2]) = I([X_1', X_2]), \quad I([X_1, X_2']) \neq I([X_1', X_2']). \tag{4}$$

This yields the contradiction with the existence of a decomposition (3). In fact, the first condition implies that $I_1(X_1) = I_1(X_1')$ and for the same reason the second one implies that $I_1(X_1) \neq I_1(X_1')$.
Now we will show that we can find such $X_1, X_1', X_2, X_2'$. We will work with $f(X) := (\alpha+\beta)\ln(1 - I(X))$ which is a monotone transform of $I$. We check that

$$f(X) = \ln W(X) - \alpha \ln \mu(X_1) - \beta \ln \mu(X_2).$$

[4]The functional form of the other index is similar $u(a,b) = -a^\alpha b^\beta,\ \alpha,\beta < 0$

The conditions (4) write as

$$\ln W(X_1, X_2) - \ln W(X_1', X_2) - \alpha\left(\ln\mu(X_1) - \ln\mu(X_1')\right) = 0,$$

$$\ln W(X_1, X_2') - \ln W(X_1', X_2') - \alpha\left(\ln\mu(X_1) - \ln\mu(X_1')\right) \neq 0.$$

Further we restrict to the case of two persons[5]

$$\frac{x_{11}^{\alpha}x_{12}^{\beta} + x_{21}^{\alpha}x_{22}^{\beta}}{x_{11}'^{\alpha}x_{12}^{\beta} + x_{21}'^{\alpha}x_{22}^{\beta}} \cdot \frac{(x_{11}' + x_{21}')^{\alpha}}{(x_{11} + x_{21})^{\alpha}} = 1; \qquad \frac{x_{11}^{\alpha}x_{12}'^{\beta} + x_{21}^{\alpha}x_{22}'^{\beta}}{x_{11}'^{\alpha}x_{12}'^{\beta} + x_{21}'^{\alpha}x_{22}'^{\beta}} \cdot \frac{(x_{11}' + x_{21}')^{\alpha}}{(x_{11} + x_{21})^{\alpha}} \neq 1$$

For simplicity of exposition we fix $x_{12} = 1, x_{22} = 2^{1/\beta}, x_{12}' = 1, x_{22}' = 100^{1/\beta}$ and $x_{11} = x_{11}' = 1$ then

$$\frac{1 + 2x_{21}^{\alpha}}{1 + 2x_{21}'^{\alpha}} \cdot \frac{(1 + x_{21}')^{\alpha}}{(1 + x_{21})^{\alpha}} = 1; \qquad \frac{1 + 100x_{21}^{\alpha}}{1 + 100x_{21}'^{\alpha}} \cdot \frac{(1 + x_{21}')^{\alpha}}{(1 + x_{21})^{\alpha}} \neq 1.$$

We also define functions

$$h_1(a, \alpha) := \frac{(1 + a)^{\alpha}}{1 + 2a^{\alpha}}, \quad h_2(a, \alpha) := \frac{(1 + a)^{\alpha}}{1 + 100a^{\alpha}}$$

Finally, the conditions write as

$$\frac{h_1(x_{21}', \alpha)}{h_1(x_{21}, \alpha)} = 1; \quad \frac{h_2(x_{21}', \alpha)}{h_2(x_{21}, \alpha)} \neq 1.$$

Now we fix $x_{21} = 1.01$. We have verified numerically that for $\alpha \in (0.02, 0.5)$[6] the equation $\frac{h_1(x_{21}', \alpha)}{h_1(1.01, \alpha)} = 1$ has a non-trivial root (i.e. different from 1) $r(\alpha)$ and checked that $\frac{h_2(r(\alpha), \alpha)}{h_2(1.01, \alpha)} \neq 1$.[7]

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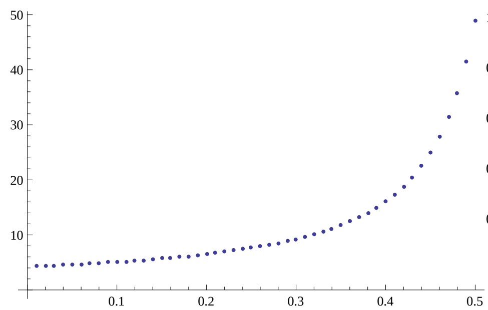
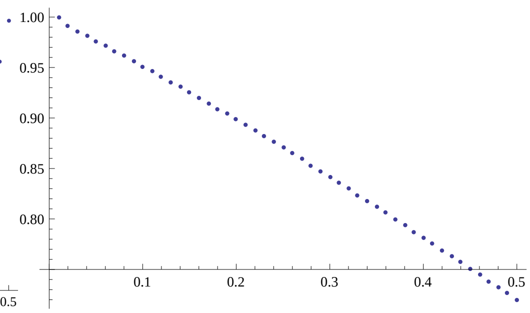
[5] This can be extended to the case of $n$ persons by giving the rest e.g. zero income.

[6] For other $\alpha$ the same procedure can be applied with different choice of $x_{12}, x_{22}$ etc.

[7] For the two matrices (rows individuals; columns attributes): $\begin{pmatrix} 1 & 1 \\ 1.01 & 2^{\frac{1}{\beta}} \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ r(\alpha) & 2^{\frac{1}{\beta}} \end{pmatrix}$ the corresponding copula $c : \{0, \frac{1}{2}, 1\}^2 \mapsto \{0, \frac{1}{2}, 1\}$ is the following: $c(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}; c(\frac{1}{2}, 1) = \frac{1}{2}; c(1, \frac{1}{2}) = \frac{1}{2}; c(1, 1) = 1$ and $c(0, x) = c(x, 0) = 0$.




(a) Non-trivial root $r(\alpha)$ of $h_1(x,\alpha) = h_1(1.01,\alpha)$




(b) Value of $\dfrac{h_2(r(\alpha),\alpha)}{h_2(1.01,\alpha)}$

# 4 Axiom and the usefulness of copulas in inequality theory

The above proof rests on step (4), which suggests the following axiom has to be fulfilled by indices decomposable into attributes via copula:
*Multivariate Separability Axiom* (MSA) For any $X_2, X_2' \in \mathbf{R}^n$ we require

$$I((X_1,X_2)) \geq I((X_1',X_2)) \wedge c(X_1,X_2) = c(X_1',X_2) \text{ iff } \\ I(X_1,X_2') \geq I(X_1',X_2') \wedge c(X_1,X_2') = c(X_1',X_2') \quad \forall_{X_1,X_1' \in \mathbf{R}^n},$$

and analogously for any $X_1, X_1' \in \mathbf{R}^n$ we require

$$I((X_1,X_2)) \geq I((X_1,X_2')) \wedge c(X_1,X_2) = c(X_1,X_2') \text{ iff } \\ I((X_1',X_2)) \geq I((X_1',X_2')) \wedge c(X_1',X_2) = c(X_1',X_2') \quad \forall_{X_2,X_2' \in \mathbf{R}^n}.$$

Axiom MSA means that the orderings imposed on any dimension when the other attribute and copula do not change is independent on the other variable or, in other words, $X_1(\text{or} X_2)$ is separable from $X_2(\text{or} X_1)$ given the copula.
The main result of NG provides for a decomposition of three bivariate indices. Clearly (2) is similar to correlation and can be interpreted as a measure of association, but it does not measure association via copula. On a deeper level, however, we wonder about the usefulness of decomposing inequality indices via copula on $n$-atomic distributions of variables that possess natural scale e.g. incomes. To see why, let us assume we decomposed the index in this way and we have two people with e.g. almost the same incomes and

we change incomes by a small amount that is enough yet to reverse the ranking. Then, relative order has changed so the value of the inequality index will change greatly, which clearly does not make much sense. This will happen except for the case when an index is defined by the functions which are defined over domains that correspond to different copulas and are "glued" in a continuous manner.[8] On the other hand, copula which is inherently scale-free, seems to be a perfect notion to measure association between ordinal variables that appear a lot in social measurement. These are usually assigned some arbitrary measure units, whereas order is in fact the only reliable information we have about them.

# References


[1] Klement E.P, Mesiar R., Logic, algebraic, analytic and probabilistic aspects of triangular norms, Amsterdam, Elsevier, 2005.

[2] Mesiar R., Discrete copulas - what they are, Joint EUSFLAT-LFA, 2005.

[3] Naga A., Geoffard P.Y, Decomposition of bivariate inequality indices by attributes, Economics Letters, Elsevier, vol. 90(3), pages 362-367, 2006.


[8] It is not clear if this can be done in a non-trivial way for more than two individuals.

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