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Variance Gamma Model in Hedging Vanilla and Exotic Options

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Abstract: The aim of this research is to explore the performance of different option pricing models in hedging the exotic options using the FX data. We analyze the narrow class of Lévy processes - the Variance Gamma process in hedging vanilla, Asian and lookback options. We pose a question of whether or not using additional level of complexity, by introducing more sophisticated models, improves the effectiveness of hedging options, assuming that hedging errors are measured as the differences between portfolio values according to the model and not real market data (which we don't have). We compare this model with its special case and the Black-Scholes model. We use the data for EURUSD currency pair assuming that option prices change according to the model (as we don't observe them directly). We use Monte Carlo methods in fitting the model's parameters. Our results are not in line with the previous literature as there are no signs of the Variance Gamma process being better than the Black-Scholes and it seems that all three models perform equally well.

Keywords: Monte Carlo, option pricing, Variance Gamma, BSM model, Lévy processes, FX market, hedging, Asian and lookback options

JEL codes: C02, C4, C14, C15, C22, C45, C53, C58, C63, G12, G13

1 Introduction

Since the publishing of the paper written by Fisher Black and Myron Scholes (Black and Scholes 1973) the idea of martingale option pricing has started. It is a way of pricing the derivative by assuming that the underlying option price is a realization of some stochastic process, then finding a probability measure under which, after discounting, the process is a martingale and then computing the option price as the discounted expected payoff of the option under this changed process. The rationale behind such a reasoning is that if that there exists a hedging strategy under real probability measure that is able to replicate the payoff with probability one, having V dollars at the beginning, then the same strategy is able to replicate it under a risk-neutral one. And since under this measure the price process is a martingale, the expected value of any strategy is equal to the value of the starting portfolio.

The idea isn't contained in the paper however, as the authors explicitly construct the hedging procedure and compute its cost via the derived PDE. The outlined approach is easier to apply in most settings as it does not require construction of a hedging scheme, although under Black-Scholes assumptions one can easily derive it for other derivatives as well - the details can be found in Lawler (2010, pp. 162-170). The method can even be applied when the hedging strategy doesn't exist, but that means that there is more than one risk-neutral measure and thus the method requires additional assumptions in this case. If possible, even a suitable PDE can be derived, similar to that of a Black and Scholes via so called Feynman-Kac formula (Lawler 2010, p. 129).

With the pricing mechanism in place we only need to find the suitable price process that fits the observed market prices. The initial choice of the Geometric Brownian Motion performs fairly well and is widely known. However it does not reflect the market reality as it assumes that the returns are normally distributed which contradicts the skewness and kurtosis we observe. Therefore since then, many other candidates have been proposed, among them one of the most popular being the Heston model (Heston 1993). The popular class of stochastic processes used in the valuation are so called Lévy processes which are defined as the processes with stationary and independent increments that are continuous in probability (Plotter 2003, p. 20) - the three desirable (though not necessarily satisfied) assumptions to have from a price process. They are however (except for the case of the Brownian Motion or a deterministic process) not continuous, which may or may not be a problem. Since they are entirely determined by the distribution of their increments (which has to be an infinitely divisible distribution) and often have readily available characteristic function, they can be easily represented via Lévy–Khintchine formula and provide option prices using for example the Fast Fourier Transform (FFT) as outlined in Tankov (2010). The processes that are of interest in this analysis belong to the class of the Lévy processes - they are called Symmetric Variance Gamma (SVG) and Variance Gamma (VG).

We pose a question of whether or not using additional level of complexity, by introducing more sophisticated models, improves the effectiveness of hedging options, assuming that hedging errors are measured as the differences between portfolio values according to the model and not real market data (which we don't have).We expect to see a result that is similar to the existing literature - the Variance Gamma outperforming both Symmetric Variance Gamma and the Black-Scholes model with the latter two not differing by much.

The paper is divided into three main parts. In the first one we review the literature on the topic of Variance Gamma option pricing and describe the data used in our analysis. In the second part we describe the options that are being hedged and the process of fitting the model, computing hedge ratios and measuring errors. In the final part we present the results, comment on them and outline the possible extensions and current limitations of this approach.

2 Literature review

Symmetric Variance Gamma Process was first introduced in Madan and Seneta (1990) and Madan and Milne (1991) initially under the name of Variance Gamma Process. The authors were able to compute its characteristic function and find the equation of the process under the risk-neutral measure. However they couldn't obtain the closed price and resorted to computing it numerically. They also compared the properties of the price obtained via SVG process to that of the Geometric Brownian Motion and noted how additional parameter might be able to capture the excess kurtosis that is present in the observed market prices and is not accounted for in the Black-Scholes model.

It was not until seven years later that the closed formula became known and was published by its authors in Madan et al. (1998). They also introduced another additional parameter to the model thus creating what is now known as the Variance Gamma Process to which the pricing formula generalizes. The additional parameter seemed to capture the skewness of the distribution. The authors also performed the empirical analysis of the models' pricing performances and compared them to that of Black-Scholes. They found that while the Symmetric Variance Gamma Process doesn't yield much improvement, the Variance Gamma Process seems to outperform both of them significantly. Unfortunately, the closed formula contained a mistake which was later corrected by Whitley (2009).

With time, more and more literature was published on the topic, both from the theoretical as well as the practical side. A way of computing VG option prices appeared in Carr and Madan (1999) by using a Fast Fourier Transform. The closed formula was known but was complicated and hard to compute, therefore the paper provided more useful way of obtaining the price. This was later improved by Chourdakis (2005) by using more flexible Fractional Fourier Transform and the associated algorithm known as Fractional Fast Fourier Transform.

Hirsa and Madan (2004) have also managed to discover a way of pricing American options within the Variance Gamma framework using the derived partial integro-differential equation. The authors in Avramidis and L'Ecuyer (2006) explored the usage of Monte Carlo and quasi-MonteCarlo methods in pricing vanilla options and examined their effectiveness.

The local version of the Variance Gamma Model has also been proposed. The relationship between the usual Variance Gamma process and its local counterpart is similar to that between Black-Scholes model and the local volatility. Since the publication of the original idea many extensions have been added to it, culminating in the recent paper by Carr and Itkin (2019).

Carr et al. (2002) created further generalization of the Variance Gamma model by adding additional parameters allowing for such features like infinite variation which they called CGMY model. Even though it is a generalization, it still remains the Lévy process. They used two estimation methods to fit the Variance Gamma parameters as well as their generalized CGMY model to the option prices for 13 different stocks. The first one, Maximum Likelihood Estimation method, was used to estimate the models' parameters under the true probability measure (similarly to what was done in Madan et al. 1998). The second one was trying to minimize the RMSE between option prices observed in the market and the ones computed by the model (similarly to what is done here) and was used to estimate models' parameters under the risk-neutral measure. In order to price the options they used FFT (Carr and Madan 1999). They observed much higher levels of skewness and kurtosis for the risk-neutral process in comparison to the statistical one. The risk-neutral process seems to exhibit infinite activity and finite variation (like Variance Gamma) but the statistical process seems to show infinite variation for some stocks. In the end they suggested building option-pricing models using completely monotone Lévy densities integrating to infinity (i.e. with infinite activity) but with finite variation. The CGMY satisfies these properties for parameter Y between 0 and 1 which is the case for the Variance Gamma process (for which $Y = 0$).

Lam et al. (2002) tested the performance of Variance Gamma and Symmetric Variance Gamma models on pricing Hang Seng Index European call options. They discovered that Variance Gamma seems to have an edge over the Geometric Brownian Motion. However it occurred that it is not very significant and that VG still suffers from a lot of biases present in the simpler model and that there is a room for improvement.

Kim and Kim (2004) made a comparison between four pricing models: Black-Scholes fitted to the implied volatility surface, the Variance Gamma, GARCH and Heston model (Heston 1993). They used data from options on KOSPI200 index. The authors concluded that the Heston model outperforms all the others both in terms of pricing and hedging performance.

In their analysis Rathgeber et al. (2016) tested different methods of estimating parameters of the Variance Gamma process using data from Dow Jones index. It expanded the work of Finlay and Seneta (2008) who had performed similar investigation using simulated data. These authors also tested several hypothesis regarding the behavior of the model's parameters under the regime-switching model developed by Hamilton (Hamilton 2010). They found that the χ^2 method described in the article provides superior performance in terms of parameter estimation. It is worth pointing out however, that these methods are of no use for our analysis as we need to estimate the process parameters under the risk-neutral measure. They have also concluded that the employment of regime-switching model helps to prevent overfitting and obtain reliable estimates.

3 Data description

We use data on EUR-USD currency pair from January 2010 to December 2019 (ten full years). It consists of pairs delta/implied volatility for butterflies and risk reversals for options on that underlying. The data was taken from Bloomberg. For the purpose of the analysis we consider five different maturities: one week, two weeks, three weeks, one month and two months. The possible (spot) deltas were (in increasing level of moneyness): 10P (-0.1 delta puts), 25P (-0.25 delta puts), ATM, 25C (0.25 delta calls) and 10C (0.1 delta calls), where ATM is defined by the strike for which the call delta is equal to the put delta (in absolute values). We use spot delta and at the money defined in this way because this is the quotation standard for the EUR-USD pair as described in Reiswich and Uwe (2012).

In Figure 1 we can see the evolution of the quotes for the analyzed instruments over the period of 10 years. Quotes for risk reversals were drifting slowly from negative values towards zero (and even were positive for a short period of time). The butterflies show opposite behavior of gradually decreasing from positive values to zero (although unlike the risk reversals the moving average never goes below 0.13).

To be able to fit our model we have to convert the information about risk reversals and butterflies to strike/price pairs for vanilla options. To do that we first extract values of implied volatility for options from the values for risk reversals and butterflies. We have:

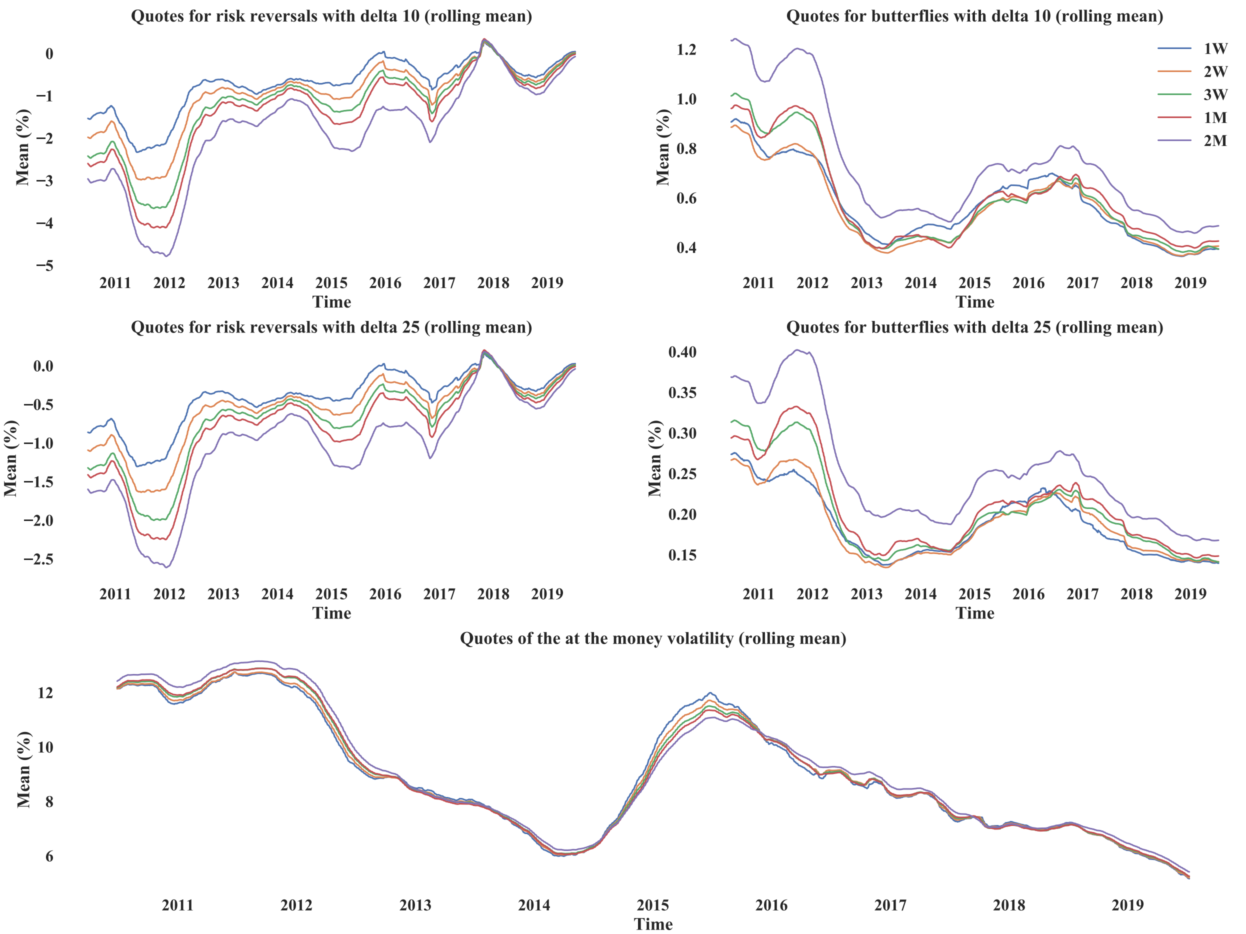
$$\sigma_{\delta,RR} = \sigma_{\delta,C} - \sigma_{\delta,P} \tag{1}$$

$$\sigma_{\delta,BF} = \frac{\sigma_{\delta,C} + \sigma_{\delta,P}}{2} - \sigma_{ATM} \tag{2}$$

by the equations (2) in Czech (2017), where $\delta \in \{10, 25\}$ in our case and:

- σ_{ATM} is the quote of the at the money volatility
- $\sigma_{\delta,RR}$ is the quote of the risk reversal volatility with delta equal to δ
- $\sigma_{\delta,BF}$ is the quote of the butterfly volatility with delta equal to δ
- $\sigma_{\delta,C}$ is the volatility of the call option with delta equal to δ
- $\sigma_{\delta,P}$ is the volatility of the put option with delta equal to $-\delta$

Figure 1. Rolling means of the quotes of butterflies, risk reversals and at the money volatility



Note: The statistics are centered and the simple moving averages are computed based on the close prices from 252 days. The figure covers the period from the middle of 2010 to the middle of 2019.

From that we can deduce:

$$\sigma_{\delta,C} = \sigma_{ATM} + \frac{1}{2}\sigma_{\delta,RR} + \sigma_{\delta,BF} \quad (3)$$

$$\sigma_{\delta,P} = \sigma_{ATM} - \frac{1}{2}\sigma_{\delta,RR} + \sigma_{\delta,BF} \quad (4)$$

which is also equation (28) in Reiswich and Uwe (2012). By applying the above transformation we obtain delta/implied volatility pairs of vanilla options, from which strikes can be deduced using delta formula from the Garman-Kohlhagen model (Garman and Kohlhagen 1983):

$$\Delta = \begin{cases} e^{-r_f\tau}\Phi(d_1) & \text{for call options} \\ -e^{-r_f\tau}\Phi(-d_1) & \text{for put options} \end{cases} \tag{5}$$

and then we can obtain prices using the pricing formula:

$$V = \begin{cases} Se^{-r_f\tau}\Phi(d_1) - Ke^{-r_f\tau}K\Phi(d_2) & \text{for call options} \\ Ke^{-r_f\tau}K\Phi(-d_2) - Se^{-r_f\tau}\Phi(-d_1) & \text{for put options} \end{cases} \tag{6}$$

where:

$$d_1 = \frac{\log(\frac{S}{K}) + (r_d - r_f + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{7}$$

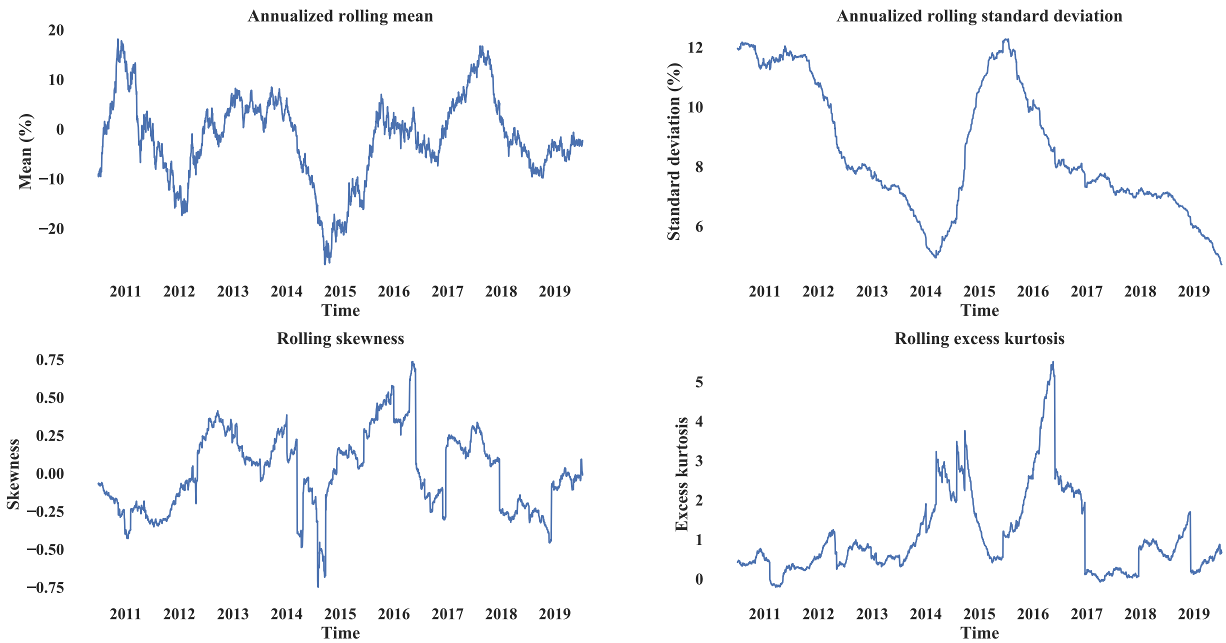
$$d_2 = \frac{\log(\frac{S}{K}) + (r_d - r_f - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{8}$$

and: S denotes the spot price, K strike, τ time to maturity, r_d and r_f denote the domestic and foreign interest rates respectively and σ denotes the volatility of the spot price.

For the spot price we use the daily close price of EUR-USD. Interest rates of both EUR and USD are taken from 10-year bonds of Germany and United States respectively. It is worth mentioning that despite risk reversals and butterflies being OTC derivatives they enjoy sufficient liquidity as the FX market is the most liquid OTC derivatives market and EUR-USD is the most popular currency pair.

As we can see in Figure 2 the data suggests that the distribution of EUR-USD returns are skewed (both positively and negatively, depending on the period) and also leptokurtic which means that Black-Scholes assumption of the price process being a Geometric Brownian Motion (Black and Scholes 1973) is not satisfied in this case.

Figure 2. Rolling statistics of EUR-USD



Note: The statistics are centered and the simple moving averages are computed based on the close prices from 252 days. The figure covers the period from the middle of 2010 to the middle of 2019.

4 Pricing models and option types

4.1 Models

We use three pricing models based on different assumptions about the underlying option price movement under the risk-neutral measure. The first price process is the Geometric BrownianMotion (GBM), the second one is the Exponential Symmetric Variance Gamma process (ESVG) and the last one is the Exponential Variance Gamma process (EVG). In each case the option price is equal to its discounted expected payoff under the risk-neutral - the martingale pricing approach we described earlier. Other assumptions are shared between the models and are the same as in Black and Scholes (1973).

There is no strike arbitrage present in the data. We haven't checked for the presence of the arbitrage between strikes and maturities as this can be difficult due to the fact, that our strike/maturity pairs are not aligned in a grid (i.e. for different maturities we have different sets of strikes).

4.1.1 Geometric Brownian Motion

Geometric Brownian Motion is the stochastic process X_t given by the following stochastic integral equation described in Kartzas and Shreve (1991, p.349):

$$X_t = X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t X_s dW_s \tag{9}$$

where W_t is a Wiener Process (Standard Brownian Motion) and the last integral is the Itô Integral (Itô 1944). Using the Itô's lemma (Kartzas and Shreve 1991, p.149) we can solve this equation to obtain a closed formula for X_t in terms of W_t:

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \tag{10}$$

If S_t is the price process which is a Geometric Brownian Motion then thanks to Girsanov theorem (Girsanov 1960) we obtain the following formula for S_t under the risk neutral measure:

$$S_t = S_0 \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \tag{11}$$

where r denotes the discount rate and q the dividend rate (or domestic and foreign rate in case of FX price). Note that in case of the Geometric Brownian motion changing to risk-neutral measure requires changing the drift μ to the cost of carry rate $b = r - q$. Having this equation we can compute the option price with maturity τ as the expected value of:

$$e^{-r\tau}\max(S_\tau - K, 0) \tag{12}$$

where K denotes the strike, thus obtaining the Black-Scholes (or Garman-Kohlhagen) formula as well as a hedging scheme that replicates this payoff with probability 1 (using the delta computed by differentiating it with respect to the spot price).

4.1.2 Exponential Variance Gamma Process

Variance Gamma Process first appeared in Madan et al. (1998) and is defined as the Brownian Motion with drift θ and standard deviation σ subordinated to a Gamma subordinator. So we can define it as the process X_t satisfying:

$$X_t = b(\gamma(t; 1, \nu)\theta; \sigma) \tag{13}$$

where $b(t; \mu, \sigma) = \mu t + \sigma W_t$ denotes the Brownian Motion with parameters μ and σ evaluated at time t and $\gamma(t; a, b)$ denotes the gamma process with parameters a and b which is defined in Applebaum (2009) as the Lévy process with increments having the gamma distribution with PDF:

$$f_{\gamma(t;a,b)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx} \tag{14}$$

where $\Gamma(x)$ denotes the gamma function. One can think of the parameter ν of the process as accountable for an excess kurtosis of the distribution and the parameter θ as responsible for the skewness (the sign of θ is the same as the sign of the skewness). Actually, looking at the moments of the distribution computed in Madan et al. (1998) one can definitely see a relationship, though its definitely more nuanced than that.

The Exponential Variance Gamma is just the exponential of the above process with (possibly) some drift added. It is important to note that we get a Geometric Brownian motion as a special case of Variance Gamma process when $\nu = 0$. However, when ν is positive the Variance Gamma is a pure jump process (and Symmetric Variance Gamma as well), so the similar reasoning as before cannot be conducted, as the Itô's lemma and Girsanov theorem can be applied only to Itô diffusions. However there still can be said something about the price process under the risk neutral measure as well as the prices and hedging of derivatives.

First of all, we can easily obtain the characteristic function of the Symmetric Variance Gamma distribution (which we define as the marginal distribution of the process) by integrating the characteristic function of marginal distribution of $b(t, \mu, \sigma)$ (which is just a normal distribution) with respect to the gamma density. It is equal to (Madan et al. 1998):

$$\phi_{X(t)}(u) = \mathbb{E}[e^{itX(t)}] = \left(\frac{1}{1 - i\theta\nu u + \frac{\sigma^2\nu}{2}} \right)^{\frac{t}{\nu}} \tag{15}$$

We can therefore obtain the value of:

$$\mathbb{E}[e^{X(t)}] = \phi_{X(t)}(-i) = \left(\frac{1}{1-\theta\nu-\frac{\sigma^2\nu}{2}}\right)^{\frac{t}{\nu}} = \exp\left(\frac{log(1-\theta\nu-\frac{\sigma^2\nu}{2})t}{\nu}\right) = e^{-\omega t} \tag{16}$$

Which means that the process:

$$Y_t = e^{X_t+\omega t} \tag{17}$$

is a process with independent increments and constant mean which obviously makes it a martingale. As in the case of the Geometric Brownian Motion the process Y_t is X_t under a different probability measure but in this case there is more than one (in fact, an infinite number of) measure changes that make e^{X_t} a martingale. This makes the approach of estimating risk-neutral Variance Gamma parameters from just the spot price impossible, because even with full knowledge of the VG parameters that govern the underlying price movements one wouldn't be able to price derivatives unless a measure change has been provided. The statement of the existence of a strategy that would replicate the payoff almost surely also doesn't hold in the presence of jumps as noticed by Merton (1976).

There are many ways to deal with the problem of non-uniqueness of the risk-neutral measure. One of them is to simply restrict yourself arbitrarily to one type of measure changes and apply now unique measure which makes a price process a martingale (after discounting). A popular choice is the Esscher transform - the details of this approach, together with its applications in option pricing can be found in Gerber and Shiu (1993).

The other way, which we will be using in this paper, is not to focus on the original price process at all but to focus on estimating parameters of the process under the risk-neutral measure directly. This approach however, requires prices of multiple derivatives on the underlying, but allows us to price and hedge derivatives without making arbitrary decisions other than choosing the pricing model itself. It is also consistent with an existing literature (An and Suo 2003; Bakshi et al. 1997).

4.1.3 Exponential Symmetric Variance Gamma Process

The SVG is just the special case of VG (but still the generalization of the BM) in which $\theta = 0$. This substitution makes all odd moments vanish and thus makes the distribution in fact symmetric. In this case the formula for the kurtosis of the distribution simplifies too and is equal to $3(1 + \nu)$, so the parameter ν can be thought of as the percentage excess kurtosis of the distribution (which matches the intuition described earlier).

4.2 Options

We use three types of options in our analysis: vanilla, Asian and lookback with fixed strike. The Asian and lookback options are based on the arithmetic mean and maximum (respectively) of daily close prices of each day between issuing of the option and its maturity (not including the day the option was issued). For consistency reasons all options are calls with five possible moneyness levels: 0.9, 0.95, 1, 1.05, 1.1. The results for put options were similar. Here we define the moneyness as the ratio of strike to the forward price. We consider 5 different maturities described earlier giving us 25 different pairs strike/maturity for each option type (so 75 options types in total). It is worth noticing that the lookbacks with moneyness 0.9, 0.95, 1 are almost always guaranteed to finish in the money, as the USD risk-free rate is almost always bigger than the EUR rate in which case the hedging procedure is exactly the same. We start our routine each week from the beginning of the 2010 to the end of 2019. There are 470 weeks in this period and thus our analysis spans across $3 \cdot 5 \cdot 5 \cdot 470 = 35250$ options in total.

5 Methodology

5.1 Parameters fitting

We fit our model parameters daily to the previously obtained strike/price pairs. They are estimated for each maturity separately, so for each day and each maturity we have 5 different strike/price pairs to fit to. We choose the values for the parameters in such a way that the root mean squared error (RMSE) between prices observed in a market and the ones predicted by the model are as small as possible. Since GBM is a special case of SVG and SVG is a special case ofVG we observe that the

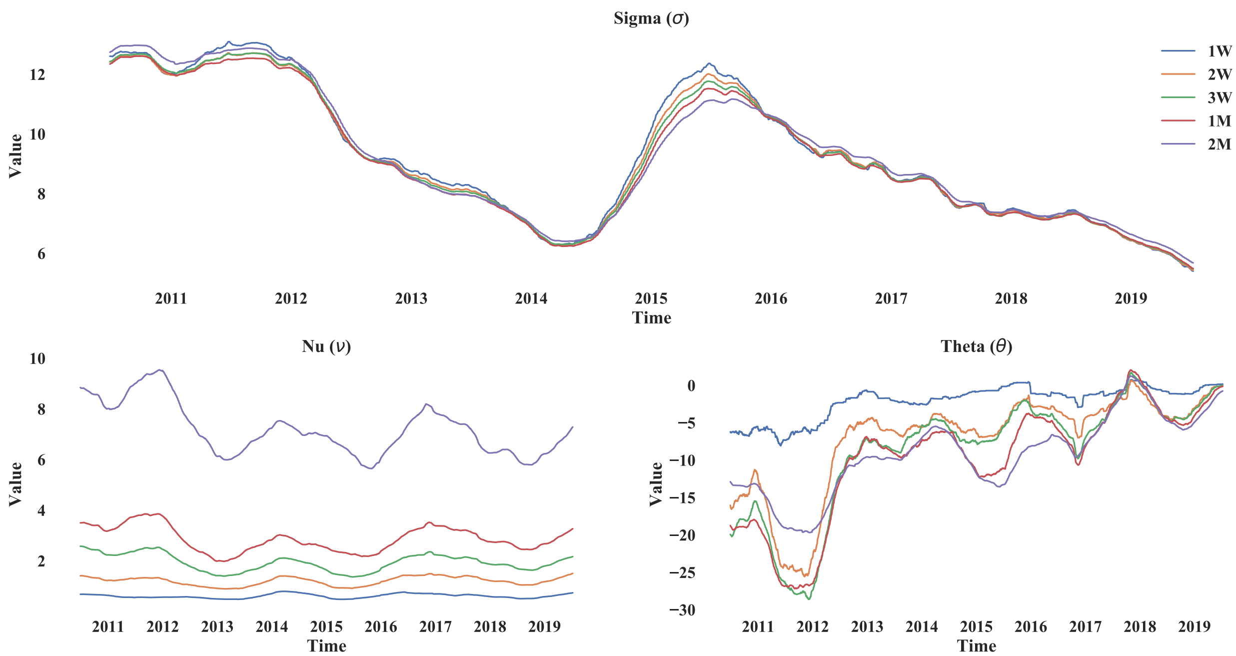
RMSE is the smallest for the VG and the biggest for the GBM Table 1. However it does not have to mean that the Variance Gamma process will be more effective in hedging derivatives.

To be able to conduct the fitting process effectively we have to be able to price vanilla options quickly. In case of the Geometric Brownian Motion we have the obvious solution in the form of Garman-Kohlhagen formula. For the Variance Gamma Process we have the closed formula in Madan et al. (1998). The problem is that it is slow and requires computation of the confluent hypergeometric function of two variables, known also as Humbert series (Humbert 1920), which has to be evaluated numerically and the literature on effective and robust computation of this function does not exist. Thus we employ a method based on numerical integration of an option price conditioned on a gamma time (which is essentially a Black-Scholes type formula, because of conditional normality) with respect to the gamma density. In order to obtain accurate results fast we use Gauss–Laguerre quadrature. Since the weights and points for which the value of the function are computed depend only on the parameter ν and time to maturity, we can price multiple strikes using the same set of points and weights which speeds up the computation. In most practical cases around 30 points is sufficient to obtain the result with double precision, however we use 50 as it doesn't slow computation by much and ensures the accuracy of the approximation. This is both faster and more accurate method than using the closed price formula or using FFT with even moderate number of points (such as $2^{11} = 2048$).

The fitting process took 30 minutes on the Intel®Core™i7-9750H Processor (base frequency 2.60 GHz with 6 cores) without the explicit use of any parallelization. The Sequential Least Squares Programming algorithm was used to minimize the objective function and the starting parameter values were taken from previously computed corresponding parameter values for simpler model (to speed up the convergence process). The remaining parameter starting value was set to 0.1.

Figure 3 illustrates that the standard deviation implied by market option prices is similar to the one obtained from the spot data (Figure 2). We can also see how the parameters ν and θ differ in time and between different maturities. It can be observed that ν is bigger for longer maturities which means that options with shorter maturities are closer to the Black-Scholes valuation but for different maturities parameters move in a similar way. This should imply that using Variance Gamma model won't make much of a difference in case of hedging shorter term options, which

Figure 3. Rolling means of parameter values for the Variance Gamma model (annualized)



Note: The statistics are centered and the simple moving averages are computed based on the close prices from 252 days. The figure covers the period from the middle of 2010 to the middle of 2019.

is magnified by the fact that there are less portfolio rebalances during the process. Parameter θ is most of the time negative which is consistent with the previous literature (for example Madan et al. 1998).

Table 1. Summary of fitting errors for different strikes and maturities

	1W	2W	3W	1M	2M
BS	0.04601	0.07568	0.10625	0.13822	0.22543
SVG	0.04069	0.07028	0.10022	0.13231	0.21704
VG	0.03552	0.03809	0.03574	0.03098	0.01281

Note: The values are the quadratic means of all 2608 errors (which are the Root Mean Squared Errors themselves) for each model and maturity.

As we can see in Table 1, although fitting errors of SVG are smaller than those of GBM hey are quite similar. On the other hand the Variance Gamma process seems to stand out with errors being couple times less than for the others (the smallest errors for each maturity are bolded).We have to be aware that it has three parameters and we are fitting it to five data points, so we are just two parameters away from complete model specification. At this point it is impossible to distinguish whether this behavior stems from better performance or just overfitting.

5.2 Hedging procedure

As mentioned before we start the hedging procedure each week and the portfolio is rebalanced daily until maturity. The method is similar to what was done in An and Suo (2003). Since it is impossible to perfectly hedge in case of SVG and VG processes we employ a minimum variance delta technique. We choose the hedge ratio that minimizes the variance of the portfolio value at the next hedging point (in our case it is the next trading day). Let us note that this delta coincides with the Black-Scholes delta only under the Black and Scholes assumptions (i.e. if underlying moves according to the Geometric Brownian Motion and the portfolio is rebalanced continuously). However, even in case of the Black-Scholes model if the portfolio is not rebalanced infinitely often the above method will produce slightly different results. That is because the minimum variance delta method will try to account for the fact that we cannot have a perfectly hedged portfolio which is caused by the lack of continuous updating of the portfolio value. The exact difference depends obviously on the hedging frequency. When the rebalancing happens as frequently as daily, the differences aren't significant.

Each day and each model we take all the options that have not yet expired and compute their minimum variance delta based on the parameters estimated on that day. More specifically, we use the parameters estimated for the earliest maturity that happens after the option expiration. So for example if we have two options, one of which expires tomorrow and the other in six days, we will use parameters estimated for the one week options in the first case and two weeks options in the second one.

There are no known closed formulas for minimum variance deltas even in cases as simple as Black-Scholes (in this case however we can represent it as an integral which can be easily calculated numerically). We can however estimate them easily thanks to the Monte Carlo simulations. Thanks to Equation (10) we can simulate from the Geometric Brownian Motion directly (that is without having to simulate the entire path and coping with discretization error). Since we can simulate from the gamma process directly as well (because its increments are independent and gamma distributed), we can simulate Exponential Variance Gamma Process and its symmetric version too, with just two simulations.

Denote by X the tomorrow stock price, by Y the tomorrow option price and by Z the final option payoff discounted δ that minimizes:

$$\text{Var}[Y - \delta X] = \delta^2 \text{Var}[X] - 2\delta \text{Cov}[X, Y] + \text{Var}[Y] \tag{18}$$

Which is just a quadratic function of δ with minimum at:

$$\delta = \frac{\text{Cov}[X, Y]}{\text{Var}[X]} \tag{19}$$

A classic hedge ratio formula. However computing δ this way seems to be feasible only for vanilla options, as only for them there are readable closed formulas for the price which allow to compute Y after sampling X. For other option types one would have to compute Y by Monte Carlo simulation again, so if we would for example want to sample 10^4 paths we would have to sample about 10^4Ys for each such simulation to achieve desired accuracy, making 10^8 simulations in total.

Fortunately, we don't have to compute Y for each value of X. If we instead sample Z (i.e.perform just one more simulation) we can obtain the estimate to the covariance by employing laws of the total expectation and covariance. From the martingale pricing approach we derive:

$$Y = \mathbb{E}[Z \mid Y] \tag{20}$$

And therefore:

$$\text{Cov}[X, Y] = \text{Cov}[\mathbb{E}[X \mid X], \mathbb{E}[Z \mid X]] = \text{Cov}[X, Z] - \mathbb{E}[\text{Cov}[X, Z \mid X]] = \text{Cov}[X, Z] \tag{21}$$

This solution makes sampling 10^4 paths feasible as we now need only $2 \cdot 10^4$ random numbers in case of GBM and $4 \cdot 10^4$ in case of SVG and VG (as we have to also sample from the gamma subordinator). This is the number of random paths that we use in the paper. It is worth pointing out that this method needs more sample paths to achieve the same accuracy because the variance of Z is greater than that of Y. However since the complexity of sampling with the first method is quadratic in the number of samples whereas the second method is linear, the latter is much more effective. It has to be mentioned that the variance of the above estimator depends heavily on the ratio between the hedging period and time to maturity. The smaller it is, the bigger the variance. Thus hedging options with longer expirations becomes more computationally expensive, especially taking into consideration that each path is longer and therefore requires sampling more points. That is why we don't include longer maturities in our analysis.

This doesn't solve the entire problem because having to hedge 35250 options, some of them for as long as 42 trading days we would have about 22 bln of random variables to sample. Therefore we introduce another optimization which boils down to sharing path simulations between options with the same type and hedged according to the same model but with different maturities. For example if we have two Asian options that we are hedging according to Variance Gamma Model, one of which matures in 6 days and the other in 7 they both need sample paths generated by the parameters estimated for the two weeks maturity (as described earlier). Therefore instead of sampling them separately, we create a single set of paths and we use the sixth day value as an estimate for Z for the first option and the seventh day for the second one.

This will introduce correlation between the errors in the estimates. In the above example if we overestimate the delta for the first option we are likely to overestimate the delta for the second option as well. But since we are using that many sample paths the errors are insignificant compared to the estimates and their correlation doesn't bother us. We should also mention that these Monte Carlo estimates can be further improved by methods such as Antithetic Variates or Control Variates (were the useful control variate could be for example the Black-Scholes delta). They are however improving the estimate by a constant factor and the asymptotic complexity remains the same. Quasi-Monte Carlo methods could potentially provide better asymptotics but for problems with such a large number of dimensions the results are unproven. The description of the mentioned methods can be found in Glasserman (2013).

The above procedure took around 90 minutes on the same, previously mentioned machine, again with no explicit use of any parallelization (but the routine can be easily done in parallel to speed up the computation).

5.3 *Error measurement*

5.3.1 *Standard method*

Since the exotic options are traded Over The Counter, we don't have pricing data at our disposal. Therefore we base our error estimation on the difference between the portfolio values on consecutive days as predicted by the model. To be able to perform our analysis we estimate two different types of errors: dollar hedging errors and absolute hedging errors. More specifically if the hedge ratio estimated at day T is h_t, the underlying and option prices at days T and $T+1$ are S_t, S_{T+1}, c_T, c_{T+1} and the annualized interest rate is r_T then the dollar and absolute hedging errors between days T and $T+1$ are equal to:

$$\epsilon_d = (c_{T+1} - h_t S_{T+1}) - e^{\frac{r_T}{252}}(c_T - h_t S_t) \tag{22}$$

$$\epsilon_a = |(c_{T+1} - h_t S_{T+1}) - e^{\frac{r_T}{252}}(c_T - h_t S_t)| \tag{23}$$

The individual day to day errors are dependent on the pricing model used but their sum will be just equal to the difference between option payoff and the final value of the portfolio and therefore is dependent on the option pricing model only via the initial price. If the pricing model is correct and we are hedged from the underlying drift we should expect that the mean of the dollar hedging errors will be zero and if the hedging procedure is effective we should observe small standard deviation.

The absolute hedging errors are based on the absolute difference between portfolio values. They are therefore not only dependent on the hedging routine but also on the pricing model and are therefore less reliable. Since each error must be positive we shouldn't expect the errors to have mean approximately equal to zero. However the proximity of zero means that both pricing mechanism and the hedging routine are correctly specified.

5.3.2 *Modified method*

The above method is very popular in research about hedging exotic options as their prices are unobservable in the market. This is for example the method employed in An and Suo (2003). However, the assumption about the option prices being the same as the ones predicted by the model is unrealistic and might yield much lower errors than the ones that would be normally observed. For example the trivial model which prices vanilla option according to its intrinsic value and has delta 1 while option is ATM and 0 while it isn't will have zero absolute errors unless the option changes its moneyness. For Asian options we can define analogous procedure but the delta in case of in the money option will be equal to the ratio between time to expiration and the entire option's lifetime. For lookback options we could have delta 1 if we are on the running maximum and 0 otherwise - this will assure zero errors unless we exceed the maximum or stop increasing after hitting one.

Therefore we propose another method of error estimation. Suppose we have some set of M models that we want to compare (in our case these are Black-Scholes, Symmetric Variance Gamma and Variance Gamma, so $M = 3$). Instead of computing portfolio's value based on the option price from the model, each day we randomly select one of the M models and compute the portfolio's value based on its option price (so there still is $\frac{1}{M}$ probability that for a given day and model the error will be the same). We try to simulate the market reality in this way, where each day one market maker with its model can have more influence on the price than the others. To avoid introducing additional randomness into the performance assessment we average the errors across all possible scenarios. Thus the modified errors will be equal to:

$$\epsilon_{d_i} = \frac{1}{M^2} \sum_{1 \leq j,k \leq M} \Big((c_{j_{T+1}} - h_{i_t} S_{T+1}) - e^{\frac{rT}{252}} (c_{k_t} - h_{i_t} S_t) \Big) \tag{24}$$

$$\epsilon_{a_i} = \frac{1}{M^2} \sum_{1 \leq j,k \leq M} |(c_{j_{T+1}} - h_{i_t} S_{T+1}) - e^{\frac{rT}{252}} (c_{k_t} - h_{i_t} S_t)| \tag{25}$$

where c_{i_t} and h_{i_t} denote the option price and estimated hedge ratio (respectively) at time T according to the i-th model. The above formula can also be thought of as the probability limit of the above mentioned Monte Carlo procedure - i.e. if we were to repeat it indefinitely then we would obtain the value from the above formula as the limit with probability one. The dollar errors can be simplified to:

$$\epsilon_{d_i} = (\bar{c}_{T+1} - h_{i_t} S_{T+1}) - e^{\frac{rT}{252}} (\bar{c}_t - h_{i_t} S_t) \tag{26}$$

where:

$$\bar{c}_t = \frac{1}{M} \sum_{j=1}^{M} c_{j_t} \tag{27}$$

is the mean of option prices from all models. The method described above still has its drawbacks.

It is very sensitive to the choice of the set of models and the results may vary and suggest that one model is better or worse depending on the choice of the models that it is compared with. If the accurate model is compared with many similar but inaccurate ones this approach will identify the accurate model as inferior as it will be further away from the consensus made by the truly inferior models. That is why this analysis should never be conducted without any additional steps taken to ensure that all the models are of sufficient quality (for example by at least checking if the model gives roughly unbiased results i.e. that its dollar hedging errors in both cases are close to zero).

6 Results

Table 2. Dollar hedging errors for Asian options with standard approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00007**	0.00229	**0.00007**	0.00228	**0.00006**	0.00229
	2W	0.00023	0.00163	0.00023	0.00163	0.00024	0.00163
	3W	0.00038	0.00138	0.00043	0.00136	0.00036	0.00134
	1M	0.00062	**0.00118**	0.00059	**0.00116**	0.00058	**0.00122**
	2M	0.00140	0.00123	0.00139	0.00128	0.00116	0.00144
ATM	1W	**0.00007**	0.00324	**0.00007**	0.00322	**0.00006**	0.00324
	2W	0.00023	0.00230	0.00023	0.00230	0.00024	0.00230
	3W	0.00038	0.00195	0.00043	0.00193	0.00035	0.00190
	1M	0.00062	**0.00168**	0.00059	**0.00164**	0.00056	**0.00181**
	2M	0.00135	0.00204	0.00106	0.00235	0.00006	0.00464
OTM	1W	**0.00007**	0.00229	**0.00007**	0.00228	**0.00006**	0.00229
	2W	0.00023	0.00163	0.00023	0.00163	0.00024	**0.00163**
	3W	0.00039	0.00138	0.00037	**0.00147**	0.00016	0.00179
	1M	0.00035	0.00134	0.00036	0.00151	0.00021	0.00202
	2M	-0.00022	**0.00121**	0.00012	0.00162	-0.00015	0.00197

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of these 470 errors.

The smallest entry for each model and moneyness has been bolded. Also the two in the money maturities as well as the two out of the money maturities have been combined into a single category for clarity.

Table 3. Dollar hedging errors for lookback options with standard approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**-0.00079**	**0.00240**	**-0.00012**	**0.00240**	-0.00019	**0.00238**
	2W	-0.00108	0.00290	0.00023	0.00296	**-0.00002**	0.00293
	3W	-0.00127	0.00327	0.00084	0.00340	0.00043	0.00370
	1M	-0.00116	0.00344	0.00145	0.00387	0.00098	0.00450
	2M	-0.00248	0.00467	0.00302	0.00565	0.00153	0.00671
ATM	1W	**-0.00080**	**0.00340**	**-0.00012**	**0.00339**	-0.00020	**0.00336**
	2W	-0.00109	0.00410	0.00022	0.00418	**-0.00003**	0.00414
	3W	-0.00128	0.00462	0.00082	0.00480	0.00042	0.00522
	1M	-0.00117	0.00485	0.00140	0.00546	0.00098	0.00634
	2M	-0.00251	0.00660	0.00288	0.00799	0.00152	0.00945
OTM	1W	**-0.00001**	**0.00009**	**-0.00002**	**0.00014**	**-0.00002**	**0.00014**
	2W	-0.00004	0.00039	-0.00009	0.00048	-0.00007	0.00042
	3W	-0.00012	0.00071	-0.00016	0.00089	-0.00009	0.00076
	1M	-0.00017	0.00105	-0.00022	0.00126	-0.00005	0.00114
	2M	-0.00079	0.00207	-0.00019	0.00260	-0.00020	0.00221

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of these 470 errors.

6.1 Dollar hedging errors with standard approach

The results are summarized in Tables 2 to 4. For Asian options in Table 2 we note that although the smallest errors are achieved for shorter maturities, the smallest standard deviation are achieved for longer ones. More specifically, in most cases we get the smallest variance for 1 month options with 2 months options being the second smallest. They also have the biggest means which might suggest that there is some bias in the hedging of the longer term options. And taking into consideration that there were 470 weeks in our analysis and therefore each entry is based on 470 we can say that these means are significantly different from zero.

This might also suggest that for smaller maturities there also is some bias but we don't have enough observations in order to show the statistical significance.

For lookback and vanilla options in Tables 3 and 4 we observe similar behavior, in case of means which are closer to zero when maturities are shorter. This is consistent with Table 1 which also showed smaller fitting errors for shorter maturities.

Table 4. Dollar hedging errors for vanilla options with standard approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00032**	**0.00026**	**0.00031**	**0.00027**	**0.00029**	**0.00028**
	2W	0.00063	0.00058	0.00058	0.00058	0.00051	0.00079
	3W	0.00090	0.00095	0.00088	0.00098	0.00051	0.00150
	1M	0.00132	0.00132	0.00112	0.00137	0.00074	0.00225
	2M	0.00233	0.00249	0.00218	0.00262	0.00042	0.00431
ATM	1W	-0.00045	**0.00249**	-0.00030	**0.00263**	**-0.00037**	**0.00260**
	2W	-0.00046	0.00277	-0.00034	0.00288	-0.00055	0.00290
	3W	-0.00044	0.00295	-0.00018	0.00313	-0.00063	0.00338
	1M	**-0.00041**	0.00291	-0.00033	0.00345	-0.00075	0.00413
	2M	-0.00093	0.00387	**-0.00002**	0.00491	-0.00169	0.00578
OTM	1W	**-0.00001**	**0.00008**	**-0.00001**	**0.00012**	**-0.00002**	**0.00012**
	2W	-0.00003	0.00029	-0.00007	0.00038	-0.00005	0.00034
	3W	-0.00008	0.00048	-0.00012	0.00068	-0.00008	0.00060
	1M	-0.00011	0.00074	-0.00020	0.00095	-0.00008	0.00082
	2M	-0.00042	0.00120	-0.00021	0.00184	-0.00034	0.00144

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of these 470 errors.

6.2 Absolute hedging errors with standard approach

The results are summarized in Tables 5 to 7. For Asian options (Table 5) it's hard to spot a clear pattern - in most of the cases the shorter maturities have the smallest errors but in some cases the longest (for OTM for GBM).

For lookbacks and vanilla options (Tables 6 and 7) we can see a clear relationship similar to what we observed in case of the dollar hedging errors - the shorter the maturity, the smaller the error. It's also worth pointing out that the errors seem to be smaller for the GBM with SVG and VG being really close to each other. The differences are small but in some cases significant. The nominal difference between GBM and SVG and VG is the biggest for at the money lookback options with maturity 2 months (Table 6) but after accounting for standard deviation we can see that it is the biggest for vanilla options with the same parameters (Table 7).

Table 5. Absolute hedging errors for Asian options with standard approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00537**	0.00180	**0.00538**	0.00182	**0.00540**	0.00183
	2W	0.00544	0.00154	0.00550	0.00161	0.00550	0.00156
	3W	0.00566	**0.00150**	0.00565	0.00153	0.00567	0.00154
	1M	0.00592	0.00154	0.00600	**0.00150**	0.00603	**0.00153**
	2M	0.00901	0.00213	0.00896	0.00217	0.00932	0.00267
ATM	1W	**0.00537**	0.00255	**0.00538**	0.00257	**0.00540**	0.00259
	2W	0.00544	0.00218	0.00550	0.00228	0.00550	0.00221
	3W	0.00566	**0.00213**	0.00565	0.00216	0.00568	**0.00219**
	1M	0.00592	0.00218	0.00601	**0.00214**	0.00615	0.00241
	2M	0.00912	0.00326	0.01007	0.00404	0.01431	0.00967
OTM	1W	0.00537	0.00180	**0.00538**	0.00182	**0.00540**	0.00183
	2W	0.00544	0.00154	0.00550	**0.00161**	0.00550	**0.00156**
	3W	0.00563	**0.00149**	0.00588	0.00170	0.00639	0.00240
	1M	0.00550	0.00193	0.00651	0.00214	0.00736	0.00361
	2M	**0.00440**	0.00222	0.00719	0.00285	0.00795	0.00505

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of the absolute values of these 470 errors.

6.3 Dollar hedging errors with modified approach

The results are summarized in Tables 8 to 10. For Asian options (Table 8) we observe similar results to the ones from Table 5 - the inverse relationship between the size of the error and its standard deviation.

For lookback and vanilla options (Tables 9 and 10) we still see that the smallest errors and their variances are achieved for the shorter maturities but the effect is not that profound as in (Tables 6 and 7).

6.4 Absolute hedging errors with modified approach

The results are summarized in Tables 11 to 13. They are very similar to what we observed in Tables 5 to 7 especially for lookback and vanilla options.

In terms of the differences between models' performances we observe that it decreased. This time both nominal and relative difference is the biggest for at the money vanilla options with ma-

Table 6. Absolute hedging errors for lookback options with standard approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00587**	**0.00232**	**0.00618**	**0.00232**	**0.00612**	**0.00227**
	2W	0.00922	0.00307	0.01027	0.00317	0.01016	0.00334
	3W	0.01210	0.00391	0.01406	0.00409	0.01393	0.00498
	1M	0.01528	0.00476	0.01831	0.00516	0.01827	0.00649
	2M	0.02796	0.00789	0.03606	0.00973	0.03476	0.01206
ATM	1W	**0.00587**	**0.00328**	**0.00617**	**0.00329**	**0.00611**	**0.00322**
	2W	0.00921	0.00436	0.01024	0.00449	0.01013	0.00474
	3W	0.01208	0.00555	0.01401	0.00581	0.01387	0.00708
	1M	0.01523	0.00676	0.01822	0.00734	0.01817	0.00923
	2M	0.02782	0.01123	0.03581	0.01386	0.03441	0.01720
OTM	1W	**0.00002**	**0.00011**	**0.00008**	**0.00019**	**0.00007**	**0.00018**
	2W	0.00023	0.00067	0.00047	0.00081	0.00038	0.00069
	3W	0.00067	0.00143	0.00132	0.00172	0.00096	0.00145
	1M	0.00149	0.00239	0.00269	0.00293	0.00188	0.00235
	2M	0.00651	0.00599	0.01100	0.00789	0.00742	0.00650

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of the absolute values of these 470 errors.

turity equal to 2 months.

6.5 Summary

The first thing that should be noticed is the fact that the results don't differ between really short maturities (less than one month). This is caused by having not enough days in the hedging process and also smaller values of ν which make the SVG and VG model closer to the Black-Scholes.

For the shorter time periods there are also no differences between strikes as both 10 and 5 percentages below or above the forward strike are so in/out of the money that the hedging procedures are identical and have deltas very close to1 1 or 0.

We can also see that the results for in/at the money lookback options are almost identical (Tables 3, 6, 9 and 12) - as we mentioned earlier, this is something to be expected as these options differ only on paper as they are guaranteed to finish in the money and hedging them is essentially equivalent to replicating the maximum value of the underlying over a period of time.

Table 7. Absolute hedging errors for vanilla options with standard approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00077**	**0.00030**	**0.00077**	**0.00031**	**0.00076**	**0.00035**
	2W	0.00207	0.00087	0.00219	0.00093	0.00237	0.00125
	3W	0.00395	0.00166	0.00413	0.00173	0.00466	0.00270
	1M	0.00657	0.00249	0.00678	0.00257	0.00801	0.00404
	2M	0.01845	0.00531	0.01928	0.00598	0.02403	0.01022
ATM	1W	**0.00390**	**0.00237**	**0.00438**	**0.00242**	**0.00432**	**0.00230**
	2W	0.00573	0.00294	0.00672	0.00305	0.00677	0.00319
	3W	0.00748	0.00369	0.00909	0.00396	0.00937	0.00509
	1M	0.00959	0.00435	0.01183	0.00473	0.01286	0.00702
	2M	0.01880	0.00796	0.02439	0.00909	0.02686	0.01481
OTM	1W	**0.00002**	**0.00009**	**0.00006**	**0.00016**	**0.00006**	**0.00015**
	2W	0.00017	0.00050	0.00037	0.00065	0.00031	0.00058
	3W	0.00045	0.00088	0.00100	0.00125	0.00075	0.00111
	1M	0.00100	0.00159	0.00200	0.00211	0.00145	0.00174
	2M	0.00415	0.00372	0.00804	0.00553	0.00582	0.00496

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of the absolute values of these 470 errors.

The overall picture of the results is that there is no improvement made by the Symmetric Variance Gamma or Variance Gamma process on hedging analyzed options. In many cases we can even observe that VG performs worse that the simpler models (in terms of absolute hedging errors) which might be a consequence of the overfitting which could also cause the significant fitting improvement visible in Table 1. However it is also hard to draw the opposite conclusion (that it is the BS model that performs better) as these differences aren't significant in most cases.

All dollar hedging errors are close to zero which means that the hedging routines are most likely asymptotically unbiased and the errors will converge to zero for sufficiently diversified portfolio of options. Any occurrence in which one model gets mean closer to 0 than the other ones should be regarded as just an accident as after accounting for the standard deviation and number of options in each category (470) the differences have no significance in all but a few cases.

It should be mentioned that even though the dollar hedging errors seem to be asymptotically unbiased they are definitely not unbiased as the hypothesis of the mean error being equal to zero can be rejected for longer maturities. For shorter maturities the bias is probably too small for test

Table 8. Dollar hedging errors for Asian options with modified approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00007**	0.00076	**0.00007**	0.00076	**0.00006**	0.00076
	2W	0.00023	0.00054	0.00024	0.00055	0.00023	0.00055
	3W	0.00038	0.00047	0.00043	0.00046	0.00036	0.00046
	1M	0.00061	**0.00041**	0.00060	**0.00041**	0.00058	**0.00043**
	2M	0.00138	0.00056	0.00137	0.00057	0.00120	0.00061
ATM	1W	**0.00007**	0.00108	**0.00007**	0.00108	**0.00006**	0.00108
	2W	0.00023	0.00077	0.00024	0.00078	0.00023	0.00077
	3W	0.00038	0.00066	0.00042	0.00065	0.00036	0.00065
	1M	0.00060	**0.00058**	0.00059	**0.00058**	0.00056	**0.00063**
	2M	0.00108	0.00167	0.00096	0.00172	0.00044	0.00216
OTM	1W	**0.00007**	0.00076	**0.00007**	0.00076	**0.00006**	0.00076
	2W	0.00023	0.00054	0.00024	0.00055	0.00023	**0.00055**
	3W	0.00033	**0.00048**	0.00036	**0.00051**	0.00023	0.00062
	1M	0.00040	0.00056	0.00035	0.00061	0.00017	0.00076
	2M	-0.00008	0.00097	0.00015	0.00103	-0.00033	0.00110

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of these 470 errors.

to be able to distinguish it from zero basing on just 470 observations.

Among all three option types that we investigate the best results seem to be achieved for the Asian options (Tables 2, 5, 7, 8, 11 and 13). This might be caused by the fact that they become easier to hedge over time as the number of unknown prices that the payoff depends on decreases and at the same time the number of known ones increases. The hardest to hedge were the lookback options (Tables 2, 5, 7, 8, 11 and 13). The difference between them and vanilla options wasn't big which should be noted as the behavior of the maximum can't be easily inferred just from the market information about the price at the end of the period (a similar thing can be said about the Asian options). The fact that the Variance Gamma provides equally good hedge might indicate that it does indeed not overfit the data as the estimated model generalizes well to more path-dependent options.

The results of our modified approach are summarized in (Tables 8 to 11 and 13). In most cases our expectations were met and the hedging errors indeed increased for all models. But the differences are similar and still there is little difference between the models' performances. All

Table 9. Dollar hedging errors for lookback options with modified approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	-0.00033	**0.00081**	-0.00035	**0.00082**	-0.00042	**0.00081**
	2W	**-0.00012**	0.00116	-0.00023	0.00118	-0.00051	0.00118
	3W	0.00022	0.00192	**0.00013**	0.00196	-0.00034	0.00202
	1M	0.00082	0.00311	0.00054	0.00317	**-0.00009**	0.00327
	2M	0.00138	0.01028	0.00140	0.01034	-0.00072	0.01041
ATM	1W	-0.00034	**0.00115**	-0.00036	**0.00115**	-0.00043	**0.00114**
	2W	**-0.00013**	0.00165	-0.00024	0.00168	-0.00052	0.00167
	3W	0.00021	0.00273	**0.00011**	0.00278	-0.00035	0.00287
	1M	0.00080	0.00444	0.00051	0.00452	**-0.00011**	0.00466
	2M	0.00130	0.01471	0.00135	0.01479	-0.00076	0.01490
OTM	1W	**-0.00002**	**0.00003**	**-0.00001**	**0.00005**	**-0.00001**	**0.00005**
	2W	-0.00006	0.00016	-0.00006	0.00018	-0.00007	0.00017
	3W	-0.00014	0.00044	-0.00009	0.00048	-0.00014	0.00046
	1M	-0.00016	0.00106	-0.00011	0.00109	-0.00018	0.00108
	2M	-0.00046	0.00601	0.00004	0.00603	-0.00077	0.00602

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of these 470 errors.

models seem to be equally affected by this modification and no model can be regarded as clearly better.

Table 10. Dollar hedging errors for vanilla options with modified approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00031**	**0.00011**	**0.00031**	**0.00011**	**0.00030**	**0.00011**
	2W	0.00059	0.00028	0.00059	0.00029	0.00054	0.00034
	3W	0.00081	0.00057	0.00085	0.00058	0.00063	0.00068
	1M	0.00116	0.00105	0.00109	0.00107	0.00092	0.00121
	2M	0.00197	0.00343	0.00198	0.00344	0.00097	0.00362
ATM	1W	-0.00038	**0.00084**	-0.00034	**0.00088**	**-0.00040**	**0.00087**
	2W	-0.00037	0.00100	-0.00039	0.00103	-0.00059	0.00103
	3W	-0.00032	0.00127	-0.00027	0.00132	-0.00067	0.00139
	1M	**-0.00029**	0.00173	-0.00041	0.00183	-0.00080	0.00198
	2M	-0.00073	0.00486	**-0.00015**	0.00496	-0.00175	0.00506
OTM	1W	**-0.00001**	**0.00003**	**-0.00001**	**0.00004**	**-0.00001**	**0.00004**
	2W	-0.00005	0.00012	-0.00005	0.00014	-0.00005	0.00014
	3W	-0.00011	0.00027	-0.00006	0.00031	-0.00011	0.00030
	1M	-0.00014	0.00054	-0.00010	0.00057	-0.00015	0.00056
	2M	-0.00043	0.00206	0.00004	0.00211	-0.00058	0.00208

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of these 470 errors.

Table 11. Absolute hedging errors for Asian options with modified approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00538**	0.00060	**0.00538**	0.00061	**0.00538**	0.00061
	2W	0.00546	0.00052	0.00550	0.00053	0.00548	0.00052
	3W	0.00567	0.00051	0.00565	0.00051	0.00565	**0.00050**
	1M	0.00597	**0.00051**	0.00599	**0.00050**	0.00599	0.00050
	2M	0.00903	0.00072	0.00899	0.00074	0.00943	0.00096
ATM	1W	**0.00538**	0.00085	**0.00538**	0.00086	**0.00538**	0.00086
	2W	0.00546	0.00073	0.00550	0.00075	0.00548	0.00074
	3W	0.00567	0.00072	0.00565	0.00072	0.00565	**0.00072**
	1M	0.00596	**0.00071**	0.00599	**0.00071**	0.00612	0.00080
	2M	0.01026	0.00174	0.01112	0.00191	0.01522	0.00349
OTM	1W	**0.00538**	0.00060	**0.00539**	0.00061	**0.00538**	0.00061
	2W	0.00546	0.00052	0.00550	**0.00053**	0.00548	**0.00052**
	3W	0.00564	**0.00051**	0.00590	0.00057	0.00642	0.00081
	1M	0.00579	0.00069	0.00659	0.00075	0.00741	0.00121
	2M	0.00595	0.00099	0.00801	0.00113	0.00837	0.00165

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of the absolute values of these 470 errors.

Table 12. Absolute hedging errors for lookback options with modified approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00603**	**0.00078**	**0.00626**	**0.00078**	**0.00620**	**0.00077**
	2W	0.01073	0.00117	0.01131	0.00119	0.01115	0.00122
	3W	0.01708	0.00194	0.01801	0.00198	0.01774	0.00214
	1M	0.02650	0.00310	0.02769	0.00315	0.02739	0.00337
	2M	0.08025	0.01015	0.08269	0.01017	0.08204	0.01038
ATM	1W	**0.00602**	**0.00111**	**0.00625**	**0.00111**	**0.00619**	**0.00109**
	2W	0.01069	0.00166	0.01127	0.00169	0.01110	0.00174
	3W	0.01701	0.00276	0.01792	0.00282	0.01764	0.00305
	1M	0.02635	0.00443	0.02751	0.00450	0.02720	0.00480
	2M	0.07956	0.01453	0.08196	0.01457	0.08125	0.01486
OTM	1W	**0.00004**	**0.00004**	**0.00008**	**0.00007**	**0.00007**	**0.00006**
	2W	0.00032	0.00023	0.00051	0.00028	0.00043	0.00024
	3W	0.00098	0.00057	0.00151	0.00068	0.00119	0.00061
	1M	0.00241	0.00123	0.00331	0.00135	0.00267	0.00123
	2M	0.01495	0.00612	0.01760	0.00623	0.01533	0.00605

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of the absolute values of these 470 errors.

Table 13. Absolute hedging errors for vanilla options with modified approach

		GBM		SVG		VG	
		Mean	Std	Mean	Std	Mean	Std
ITM	1W	**0.00075**	**0.00010**	**0.00077**	**0.00010**	**0.00077**	**0.00012**
	2W	0.00215	0.00032	0.00225	0.00034	0.00239	0.00045
	3W	0.00419	0.00066	0.00434	0.00069	0.00485	0.00098
	1M	0.00727	0.00112	0.00744	0.00116	0.00853	0.00156
	2M	0.02325	0.00340	0.02395	0.00350	0.02800	0.00443
ATM	1W	**0.00403**	**0.00080**	**0.00432**	**0.00080**	**0.00427**	**0.00076**
	2W	0.00604	0.00104	0.00668	0.00101	0.00666	0.00102
	3W	0.00815	0.00136	0.00923	0.00139	0.00933	0.00163
	1M	0.01094	0.00175	0.01251	0.00181	0.01299	0.00225
	2M	0.02519	0.00395	0.02865	0.00415	0.03020	0.00509
OTM	1W	**0.00003**	**0.00004**	**0.00007**	**0.00006**	**0.00006**	**0.00005**
	2W	0.00025	0.00018	0.00041	0.00022	0.00034	0.00020
	3W	0.00072	0.00036	0.00115	0.00047	0.00091	0.00042
	1M	0.00163	0.00069	0.00239	0.00083	0.00191	0.00071
	2M	0.00798	0.00223	0.01063	0.00256	0.00894	0.00238

Note: Hedging errors were computed using hedge ratios and option prices from a Monte Carlo simulation with 104 trials with parameters estimated by minimizing the SSE between the prices observed in a market and the ones predicted by a model. The above values are means and standard deviations of the absolute values of these 470 errors.

7 Conclusion

The aim of this research was to explore the performance of different option pricing models in hedging the exotic options using the FX data. Since the exotic options' quotes are in most cases impossible to obtain it is hard to conduct research and produce any result without making some unrealistic assumptions. Therefore the topic seems to be underexplored compared to the vanilla options research (although the market for exotic options is also much smaller). We proposed the method of estimating their hedge ratios as well as the simple framework that could be helpful in gauging their performance.

We have used the data for the EURUSD currency pair for years 2010-2020. More specifically, we have taken the daily quotes for the risk reversals and butterflies with deltas 10 and 25 from which we have extracted the implied volatilities for vanilla options, their strikes and prices. This allowed us to fit the models' parameters in order to minimize the SSE between the observed prices and the ones predicted by the model. Then we proceeded to compute hedge ratios for vanilla and exotic options of our choice using minimum variance delta technique in conjunction with Monte Carlo estimation. This in turn led us to compute the hedging errors of our procedure. Since we didn't have the daily quotes for the option prices that we were hedging we have made the assumption that they are changing as predicted by the model. Since this could lead to the underestimation of the real error size we have proposed another method which aims at mitigating this bias.

The results are not in line with the previous literature as there are no signs of the Variance Gamma process being better than the Black-Scholes and it seems that all three models perform equally well. Previous results were however concerning the pricing performance and our approach focuses on the minimum variance hedging. The underlyings and the time periods also differ. The latter might be even more important as since the early 2000s the market reality changed significantly. The most important difference in results comes from the methodology of fitting parameters, computing hedge ratios as well as measuring errors. Since the SVG and VG models haven't showed any superior performance it seems that the best model to use under these assumptions would be the BSM model as it achieves not worse (if not just better) results without being that computationally intensive.

It is also important to stress the limitations of this approach. Since we are hedging just using a single instrument - the underlying, there is little room for further improvement and there is a limit (unfortunately unknown) on how accurate results one can obtain with such a limited class of hedging instruments. This makes it even harder to asses performance especially taking into consideration the fact, that despite its limitations and incorrect assumptions, the Geometric Brownian Motion seemed to be a relatively good fit providing dollar errors with means close to zero.

The method of estimating minimum variance delta using Monte Carlo simulations can be further expanded by considering more than one hedging instrument. In case we are allowed to trade in n different instruments with tomorrow prices $X = (X_1, \dots, X_n)$ and the tomorrow option price is Y with its payoff (discounted to tomorrow) Z we obtain a formula similar to Equation (18):

$$\text{Var}[Y - \delta X] = \delta^T \Sigma_{XX} \delta - 2\delta^T \Sigma_{XY} + \text{Var}[Y] \tag{28}$$

Where δ is a n-dimensional vector of hedge rations and Σ_{ab} denotes the covariance matrix between vectors a and b. Minimizing the above expression with respect to δ yields the generalization of Equation (19):

$$\delta = \Sigma_{XX}^{-1} \Sigma_{XY} = \Sigma_{X}^{-1} \Sigma_{XZ} \tag{29}$$

Where the last equality can be proven similarly to what was done in Equation (21). Then we can again estimate both the matrix Σ_{XX}^{-1} and the vector Σ_{XY} using Monte-Carlo techniques. However for this method to work we need to be able to sample from the joint distribution of $(Y, X_1, \dots, X_n)$ or $(Z, X_1, \dots, X_n)$ (the latter is most likely much simpler). If X_i are simple derivatives on the underlying and can be priced exactly, the easiest way to achieve that is to firstly sample the underlying and then proceed to price all other instruments accordingly (and sample Z conditional on the underlying price). Unfortunately the method is not applicable in cases where X_i are more complicated and can't be easily priced according to the model.

The approach can also be extended to other options. But the requirement of being able to sample from the joint distribution of $(Z, X_1, \dots, X_n)$ remains which means that in particular if we want to do that by sampling this vector based on the next period spot price, we have to have an easy way of sampling Z. This is of course trivial in case of the vanilla options, little harder in case of lookback and Asian options but can be difficult in case of, for example, compound options. But even for them we can replace the payoff, which in this case is an underlying option price, by the discounted payoff of the underlying option - by the similar reasoning as in Equation (21) we can prove that it will yield the same covariance and price using Monte Carlo simulation as computing the payoff (i.e. the option price) directly. So as we see the method described in this paper can be very flexible.

The modified approach also can be further expanded as different models might have different weights (i.e. probabilities of choosing them as the pricing model at any given day). One can even consider an iterative procedure in which the better models get bigger weights in the next iteration (although one has to be cautious not to increase the weights of the worse performing models as it will create a negative feedback loop). One can even introduce dependencies between the probabilities on consecutive days and make the model more likely to remain the pricing model for the next day (which we can even condition on the performance of this model at that day).

The pricing model can be arbitrary as long as there is a convenient way of sampling from the marginal distributions of its price paths under risk-neutral distribution so for example all Lévy models seem to be suitable for such analysis.

References

An, Yunbi and Wulin Suo. 2003. *The Performance of Option Pricing Models on Hedging Exotic Options*. Tech. rep. Working Paper.

Applebaum, David. 2009. *Lévy processes and stochastic calculus*. Cambridge university press.

Avramidis, Athanassios N and Pierre L'Ecuyer. 2006. "Efficient Monte Carlo and quasi–Monte Carlo option pricing under the variance gamma model". *Management Science* 52.12, pp. 1930–1944.

Bakshi, Gurdip, Charles Cao, and Zhiwu Chen. 1997. "Empirical performance of alternative option pricing models". *The Journal of finance* 52.5, pp. 2003–2049.

Black, Fischer and Myron Scholes. 1973. "The pricing of options and corporate liabilities". *Journal of political economy* 81.3, pp. 637–654.

Carr, Peter, Hélyette Geman, Dilip B Madan, and Marc Yor. 2002. "The fine structure of asset returns: An empirical investigation". *The Journal of Business* 75.2, pp. 305–332.

Carr, Peter and Andrey Itkin. 2019. "Geometric Local Variance Gamma Model". *The Journal of Derivatives* 27.2, pp. 7–30.

Carr, Peter and Dilip Madan. 1999. "Option valuation using the fast Fourier transform". *Journal of computational finance* 2.4, pp. 61–73.

Chourdakis, Kyriakos. 2005. "Option pricing using the fractional FFT". *Journal of Computational Finance* 8.2, pp. 1–18.

Czech, Katarzyna. 2017. "Extracting Market Expectations from Currency Options' Risk Reversals". *Annales Universitatis Mariae Curie-Skłodowska, Sectio H Oeconomia* 51.6, pp. 63–73.

Finlay, Richard and Eugene Seneta. 2008. "Stationary-Increment Variance-Gamma and t Models: Simulation and Parameter Estimation". *International Statistical Review* 76.2, pp. 167–186.

Garman, Mark B and Steven W Kohlhagen. 1983. "Foreign currency option values". *Journal of international Money and Finance* 2.3, pp. 231–237.

Gerber, Hans U and Elias SW Shiu. 1993. *Option pricing by Esscher transforms*. HEC Ecole des hautes études commerciales.

Girsanov, Igor Vladimirovich. 1960. "On transforming a certain class of stochastic processes by absolutely continuous substitution of measures". *Theory of Probability & Its Applications* 5.3, pp. 285–301.

Glasserman, Paul. 2013. *Monte Carlo methods in financial engineering*. Vol. 53. Springer Science & Business Media.

Hamilton, James D. 2010. "Regime switching models". *Macroeconometrics and time series analysis*. Springer, pp. 202–209.

Heston, Steven L. 1993. "A closed-form solution for options with stochastic volatility with applications to bond and currency options". *The review of financial studies* 6.2, pp. 327–343.

Hirsa, Ali and Dilip B Madan. 2004. "Pricing American options under variance gamma". *Journal of Computational Finance* 7.2, pp. 63–80.

Humbert, P. 1920. "Sur les fonctions hypercylindriques". *Comptes Rendus Hebdomadaires Séances l'Acad. Sci., Paris* 171, pp. 490–492.

Itô, Kiyosi. 1944. "109. stochastic integral". *Proceedings of the Imperial Academy* 20.8, pp. 519–524.

Kartzas, I and SE Shreve. 1991. *Brownian Motion and Stochastic Calculus (Graduate Texts in Mathematics)*.

Kim, In Joon and Sol Kim. 2004. "Empirical comparison of alternative stochastic volatility option pricing models: Evidence from Korean KOSPI 200 index options market". *Pacific-Basin Finance Journal* 12.2, pp. 117–142.

Lam, K, Eric Chang, and Matthew C Lee. 2002. "An empirical test of the variance gamma option pricing model". *Pacific-Basin Finance Journal* 10.3, pp. 267–285.

Lawler, Gregory F. 2010. "Stochastic calculus: An introduction with applications". *American Mathematical Society*.

Madan, Dilip B, Peter P Carr, and Eric C Chang. 1998. "The variance gamma process and option pricing". *Review of Finance* 2.1, pp. 79–105.

Madan, Dilip B and Frank Milne. 1991. "Option pricing with vg martingale components 1". *Mathematical finance* 1.4, pp. 39–55.

Madan, Dilip B and Eugene Seneta. 1990. "The variance gamma (VG) model for share market returns". *Journal of business*, pp. 511–524.

Merton, Robert C. 1976. "Option pricing when underlying stock returns are discontinuous". *Journal of financial economics* 3.1-2, pp. 125–144.

Plotter, PE. 2003. "Stochastic Integration and Differential Equation". *Stochastic Modeling and Applied Probability* 21.

Rathgeber, Andreas W, Johannes Stadler, and Stefan Stöckl. 2016. "Modeling share returns-an empirical study on the Variance Gamma model". *Journal of Economics and Finance* 40.4, pp. 653–682.

Reiswich, Dimitri and Wystup Uwe. 2012. "FX volatility smile construction". *Wilmott* 2012.60, pp. 58–69.

Tankov, Peter. 2010. "Financial modeling with Lévy processes".

Whitley, Alan. 2009. "Pricing of European, Bermudan and American Options under the Exponential Variance Gamma Process".

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