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Value-at-risk — the comparison of state-of-the-art models on various assets

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Abstract: This paper compares different approaches to Value-at-Risk measurement based on parametric and non-parametric approaches. Three portfolios are taken into consideration — the first one containing only stocks from the London Stock Exchange, the second one based on different assets of various origins and the third one consisting of cryptocurrencies. Data used cover the period of more than 20y. In the empirical part of the study, parametric methods based on mean-variance framework are compared with GARCH(1,1) and EGARCH(1,1) models. Different assumptions concerning returns' distribution are taken into consideration. Adjustment for the fat tails effect is made by using Student t distribution in the analysis. One-day-ahead 95%VaR estimation is then calculated. Thereafter, models are validated using Kupiec and Christoffersen tests and Monte Carlo Simulation for reliable verification of the hypotheses. The overall goal of this paper is to establish if analyzed models accurately estimate Value-at-Risk measure, especially if we take into account assets with various returns distribution characteristics.

Keywords: risk management, Value-at-Risk, GARCH models, returns distribution, Monte Carlo Simulation, asset class, cryptocurrencies

JEL codes: C4, C14, C45, C53, C58, G13

Note: The views presented in this text are those of the authors and do not necessarily represent those of Labyrinth HF project.

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1 Introduction

The term Value-at-Risk can be traced to the Group of Thirty[1] report (G-30 report, 1993) published in July 1993. It was at this moment, risk management in financial institutions had entered a new era. Today, after a quarter of a century and immense progress in the development of numerous analytical tools for measuring and managing risk, Value-at-Risk models are still a solid foundation to build upon. In January 2019, the Basel Committee on Banking Supervision — one of the BIS[2] committees — published an updated standard of „Minimum capital requirements for market risk" (BCBS report, 2019). This framework heavily depends on the Value-at-Risk measurement, which shows that this relatively simple concept, despite many faults such as serious underestimation of risk during times of extremely high volatility (Žiković and Aktan, 2009), is still a valuable tool in managing assets of an institution.

The main goal of this paper is to examine which methods of VaR calculation underestimate or overestimate the market risk of a portfolio. Four different approaches are analyzed — Historical Simulation, Variance-Covariance approach and two univariate ARCH-class models: GARCH(1,1) and EGARCH(1,1). Secondly, for the last three methods calculations are made twice: assuming normally-distributed portfolio returns and assuming that these returns are distributed according to the generalized t distribution. This is to determine whether or not changing assumptions about the returns' distributions give statistically different results. In general, one might expect that models using student t distribution are better fitted to the data. Nevertheless, this does not necessarily mean they provide more accurate forecasts. Three independent portfolios are taken into consideration. The first one consists only of stocks from the London Stock Exchange, the second one is a diversified portfolio of assets from several markets and the third is portfolio of cryptocurrencies. This discrepancy is intended to help with an assessment of the influence of correlation between assets on the estimation of a portfolio's risk using seven aforementioned approaches.

The main hypothesis of this paper is that GARCH(1,1) with assumption of normally-distributed portfolio returns is the best choice for forecasting daily 95% Value-at-Risk. Returns on the financial markets tend to have student t, not gaussian distribution. However,

[1]Group of thirty is an international group of remarkable academics and leading specialists from the financial industry. It was founded in 1978, currently under the leadership of Paul Volcker — an American Economist and former Chair of the Federal Reserve. Its main goal is to „deepen understanding of international economic and financial issues, and to explore the international repercussions of decisions taken in the public and private sectors" as stated on the official website. Source: https://group30.org/about

[2]Bank for International Settlements, founded in 1930 as a consequence of Young Plan realization. Located in Basel, nowadays BIS hosts high-level meetings and supports international cooperation amongst central banks via the so-called Basel Process. Source: https://www.bis.org/about/index.htm?m=1%7C1

in forecasting VaR measurement using t-student distribution may lead to underestimating risk. Simultaneously, using ARCH models for estimating VaR is a more complex way than the basic approaches. Therefore, the latter ones should be more conservative in forecasting risk. Next notion is that mean-variance framework produces better forecasts for more diversified portfolios. If our portfolio consists of various types of assets, including additional information about their interaction may improve our model. The last hypothesis is that using the same method in estimating VaR gives significantly different results depending on the class of assets in the portfolio.

First chapter describes the motivation behind writing this paper and discusses briefly the history of risk management and the most important issues concerning this subject. Methods of risk measurement beyond Value-at-Risk are shortly mentioned. Afterwards, literature concerning models used in a study is reviewed. Some groundbreaking articles relating to forecasting volatility using ARCH-class models are referred, as well as other papers of more empirical character. Moreover, stylized facts about volatility are presented and reasons behind the choice of the methodology are revealed. The next chapter consists of the theoretical background behind the empirical analysis. Firstly, Value-at-Risk measure is properly defined and discussed. Chosen models' formulae and equations used in calculations of VaR are explored thoroughly. The third chapter discusses in detail data used in the analysis. Components of portfolios are shown, as well as their main statistical features. Following section consists of empirical analysis. Number of results are presented, then models are verified by the use of several backtesting procedures. Finally, the last chapter contains an overview of the whole analysis. Concluding remarks are made within the summary.

2 Motivation and a brief history of risk management practices

The modern history of risk management started with Markowitz's seminal paper on portfolio selection (Markowitz, 1952). Next steps in the development of risk management tools came with works of William F. Sharpe (Sharpe 1963), Fischer Black, and Myron S. Scholes (Black and Scholes, 1972, 1973). Since then, financial econometrics has developed substantially and nowadays is a vast branch of economics. Continuously increasing size of financial markets observed in the last couple of decades forced economists to develop new ways of dealing with high volatility. In the late 1980s and early 1990s, risk management and volatility modeling methods thrived in economic journals (Engle, 1982; Bollerslev, 1986, 1990; Nelson, 1991; Zakoian, 1994, etc.). However public understanding of different kinds of risk and volatility measures was only superficial. Recent studies of Daniel Goldstein and Nassim Taleb (Goldstein and Taleb, 2007) suggest this trend may continue today. To support this claim one might consider series of financial fiaschi of many companies around the world in market risk management[3]. After the events of October 1987, it became clear that solid foundations in the risk management area should lie at the very core of all institutions in the financial world. In order to survive market turmoils firms had to find a precise and efficient way to determine the risk of their assets on a daily basis. In response to those pressing matters, JPMorgan introduced RiskMetrics™ in 1994 (JP Morgan, 1996) setting a new industry standard in risk management and popularizing Value-at-Risk concept. Intuitively VaR is defined as a potential loss, such that the probability of its exceedance over a time horizon is equal to some previously specified level of significance (Kuziak, 2003). Hence, it is a function of two variables, the choice of which depends on risk manager and should be conducted very carefully. Usually, it is determined by a variety of factors such as: types of assets, main goal of the analysis, availability of information or constraints enforced by regulatory agencies. Amongst many others, good risk management practices require a transparent measure of risk and rigorous measurement techniques. These criteria were met by RiskMetrics™ (JP Morgan, 1995), though this methodology was not without flaws. At the time of 2008 financial crisis, many found out that during extreme market conditions, Value-at-Risk (and subsequently RiskMetrics™ methodology) underestimates risk dramatically (Oanea and Anghelache, 2008). Before those events, back in 1998, the theory of coherent measures of risk were introduced in the academic community (Artzner et al. 1997; 1998). It was the first set of axioms based on mathematical foundations for risk measurements. As stated in Dowd (2005), measurement $\rho(.)$ is coherent if and only if, for two portfolios' Profit/Loss X and Y satisfies conditions I - IV:

I *Monotonicity: If* $Y \geq X$ *then* $\rho(X) \leq \rho(X)$

II *Subadditivity:* $\rho(X+Y) \leq \rho(X) + \rho(Y)$

III *Positive homogenity:* $\rho(hX) = h\rho(X)$ *, for* $h > 0$

IV *Translational invariance:* $\rho(X+n) = \rho(X) - n$ *, for some amount of* n*.*

Dowd specifies also, that VaR is subadditive only if portfolio's P/L is elliptically distributed. This is a major problem of VaR measure because, beside the rare case of ellipticality of returns, it does not satisfy the second condition. In which case, risk-return analysis is not consistent with von Neumann-Morgenstern expected utility theory (Neumann and Morgenstern, 1953). The answer to these problems may lay in second widely spread risk measure, the Expected Shortfall. The ES is defined as the average of the worst $\alpha\%$ of losses. Unlike

[3]Showa Shell in 1993, Metallgesellschaft in 1994, Barings in 1995 and Procter&Gamble in 1994 (Jorion, 2002)

VaR, it is a coherent risk measure, and it allows to determine risk more precisely during times of increased volatility. Nowadays, many other risk measures have been developed. However, despite its drawbacks, VaR is still a foundation to build upon in modern risk theory. Not only can it be used in more sophisticated models but it is widely used in its pure form (along with the ES) due to regulatory restrictions and easy implementation.

Calculation of Value-at-Risk is mostly dependent on accurate volatility estimation. While RiskMetrics™ was becoming more popular, other ways of its calculation were being developed. Its methodology is the special case of the IGARCH(1,1) model (Buła, 2016) which belongs to the ARCH-class models started by Robert Engle (Engle, 1982) and later developed by Tim Bollerslev (1986). These were the first attempts to account for the volatility clustering phenomenon (Mandelbrot, 1963) while estimating volatility. Later on, the multivariate approach to GARCH modeling was developed (Bollerslev, 1990). Number of stylized facts about volatility behavior can be named (Piontek, 2003):

- *volatility clustering* effect
- „*thick tails*" of returns distribution function
- *skewness* of the rate of returns' distribution
- *autocorrelation* of returns
- *leverage* effect
- *long memory* of the returns
- *long memory* of the volatility

In order to consider these effects in volatility modeling, different attempts were made. All of the ARCH models can explain „volatility clustering"and „thick tails"effects. Exponential GARCH (Nelson, 1991) and Threshold ARCH (Zakoian, 1994) answer the problem of the „leverage effect", which is the asymmetry in returns' volatility. Assuming different distribution in GARCH process (Bollerslev 1987), for example generalized student distribution, may resolve the skewness problem. While APRACH model (Ding, 2011) additionally considers long-memory property of returns, models such as FIGARCH (Baillie et al. 1996) take into account long memory of the volatility. It is possible to create even more complicated models such as Fractionally-Integrated-APARCH model which consider all of the volatility's properties mentioned above[4]. Aside from the ARCH-models, other efforts to improve the accuracy

[4] For more on different ARCH-class models and volatility characteristics they describe, see 2003 review by Ser-Huang Poon and Clive W. Granger. (Poon and Granger, 2003)

of volatility estimates were made. Realized volatility concept was introduced (Andersen et al. 2001; 2003) to include intraday data into evaluating daily volatility. Although, Pierre Giot and Sebastien Laurent (Giot and Laurent, 2003) showed that ARCH type models can deliver accurate VaR forecasts, as long as such model takes into account statistical characteristics of the data (for example they implement skewed Student APARCH model to resolve the issue of high kurtosis and skewness). Amongst many other approaches to estimating VaR, two others deserve additional mentioning. One of them is the Quantile Regression Approach, a good example of which is Conditional Autoregressive Value at Risk model, so-called CAViaR model (Engle and Manganelli 2004). Another one is the estimation of VaR and ES via Copula functions (Papla and Piontek, 2009). Copulas allow modeling interactions and dependence structure in multivariate model cases.

It is worth mentioning that methods and models developed to forecast volatility are not only of an academic inquiry but also of a practical nature. Therefore optimal VaR estimation technique should not only be accurate but also straightforward and easily applied in real situations. Hence, if one model turns out to be more precise than other, but the difference turns out to be almost not statistically significant (despite better economic grounds), we shall choose the simpler model. Such solution ensures, that cost of implementation of more complex model (for example while calculating Value-at-Risk of an enormous portfolio) will not exceed the profits obtained due to accuracy. Moreover, Fiszeder (2007) shows that using additional information not necessarily leads to better forecasts of volatility estimators. Having that in mind the authors of this paper decided to choose two ARCH-class models for the analysis: GARCH(1,1) and EGARCH(1,1). The main goal of this paper is to compare relatively straightforward methods of forecasting VaR of a portfolio. Keeping models simple will allow to avoid overfitting of the models to the data and achieve reliable results for VaR estimation. Generalized t distribution should also contribute to increased precision of models as many studies show that it can be used to help with skewness and asymmetry of the distribution of data.

3 Models — theoretical background

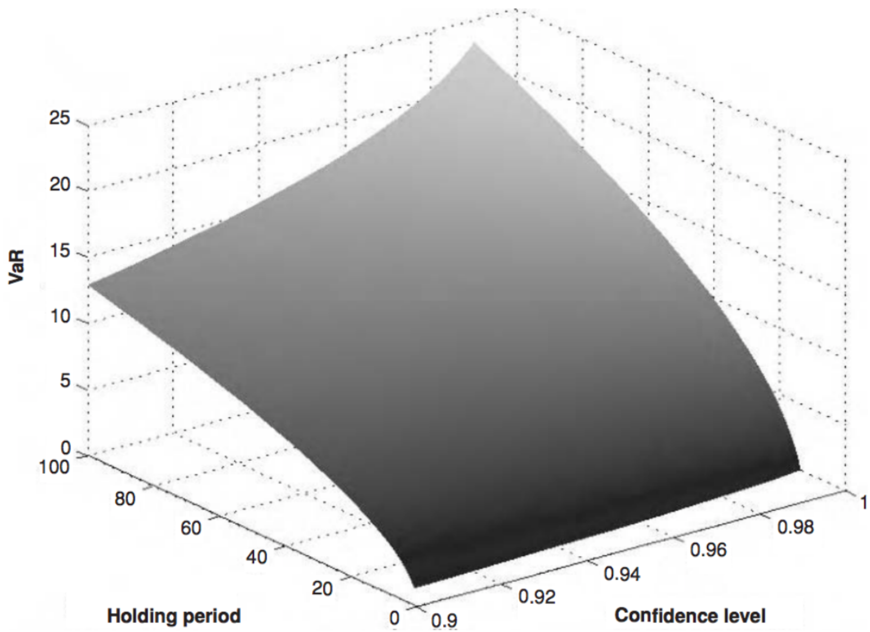
3.1 Value at Risk — definition

Mathematically speaking, Value-at-risk is a negative quantile of (assumed over some set horizon) Profit/Loss distribution, given some confidence level. From Dowd (2005) let p be a confidence level and α significance level, where $p = 1-\alpha$. If q_α is the α-quantile of portfolio's P/L over some holding period h, then Value-at-Risk of this portfolio is equal to:

$$VaR = -q_\alpha \quad (1)$$

To illustrate this concept, let's assume that returns of an asset over some holding period h are normally distributed. Then, to calculate VaR one has to calculate appropriate quantile. For example 95%VaR is equal to negative quantile $q_{0.05}$. Therefore, 95%VaR ≈ 1.645.

Figure 1: A VaR surface as a function of two parameters



Note: Chart produced using normalvarplot3D function. It illustrates a hypothetical portfolio's VaR estimates (vertical axis) at different confidence level and holding period, assuming $\mu = 0$ and $\sigma = 1$. Source: „Measuring market risk" Kevin Dowd.

As one may notice, VaR measurement of a portfolio is a function of two arbitrarily chosen variables: h and p. Dowd (2005) illustrates how this measurement changes in respect to each variable, assuming normally-distributed portfolio returns. It can be observed in Figure 1, that as confidence level and holding period increase, VaR measure rises as well. Hence, the most crucial part of evaluating VaR of a portfolio starts with careful selection of these two parameters. Using only one point from the surface to represent risk of a portfolio, deprives the risk manager of information about it. The selection of a holding period depends on two aspects: the liquidity of the assets and the size of the data set used for backtesting procedures. If portfolio consists entirely of assets that might take a couple of days to sell, choosing $h = 1$ day might be unwise. There is a chance that by the time they are liquidated, risk will have changed significantly, rendering whole analysis obsolete. As for the size of the data used, the shorter holding period is preferred. Let's assume for example one has at disposal period of 1000 observation and chooses to estimate 99%VaR for 10 days holding period. Accurately calculated 99%VaR should underestimate risk in 1% of cases. This means that actual P/L exceeds VaR only once during approximately 4 years of the analyzed period. Such data sample is most certainly not big enough for statistical inference. The choice of the confidence level depends mainly on the purpose of analysis. Lower confidence level can be more useful in backtesting procedures for the same reason as shorter holding period was. Typically p equals to 90%, 95% or 99%. RiskMetrics^{TM} methodology assumes $p = 95\%$ over $h = 1$ day. However, BCBS market risk framework (2019) for calculating minimum capital states that „Backtesting of the bank-wide risk model must be based on a VaR measure calibrated at a 99th percentile confidence level." Fortunately, different conversion methods can be applied to go through different points on the VaR surface without the necessity to calculate it all over again. Philippe Jorion (2002) provides an example how to convert RiskMetrics^{TM} 95%VaR over 1 day ($\mathrm{VaR}_{\mathrm{RM}}$) to The Basel Committee 99%VaR over 10 days ($\mathrm{VaR}_{\mathrm{BC}}$):

$$VaR_{BC} = VaR_{RM} \frac{2.33}{1.65} \sqrt{10} = 4.45 \, VaR_{RM} \tag{2}$$

This analysis is based on the assumption that holding period is equal to 1 day, and confidence level is set to 95%. Such choice of the parameters assures that sufficient data for backtesting procedures, for capital requirements will not be calculated directly. Time horizon corresponds also to the fact that there is no problem in liquidating assets daily in either portfolio. In the next subsections, different methods for calculating VaR are presented. First of them is simple Historical Simulation. Portfolios' returns are calculated and based on their empirical distribution, appropriate quantile is computed. Next, the Variance-Covariance approach is discussed, assuming both normal, and generalized student t distribution of portfolios' returns. After that, two univariate ARCH-class models are examined. Both GARCH(1,1) and EGARCH(1,1) are estimated given two specifications: assuming normal and student distributions of returns.

3.2 Historical Simulation

To calculate VaR using Historical Simulation method one first has to define formally what portfolio rate of returns is. Daily rate of return of an asset i at the time t is defined according to the formula:

$$r_{i,t} = ln\left(\frac{p_{i,t}}{p_{i,t-1}}\right) \tag{3}$$

Where $p_{i,t}$ denotes adjusted price of an asset i on day t. Having returns of assets in the portfolio, simple arithmetic mean is calculated to obtain portfolio's rate of return. Expressing VaR as a percentage of the portfolio's value, let's define 95% 1-day historical VaR as 0.05 quantile of an empirical return distribution (Alexander, 2008). Therefore we obtain:

$$VaR_{HS} = -q_{0.05}\left(\overrightarrow{r}_{P/L}\right) \tag{4}$$

Were $\overrightarrow{r}_{P/L}$ represents a vector of portfolio's actual Profit/Loss rates of returns for last n observations. Time series of VaR values is obtained by calculating VaR using historical data from the last 252 observations according to formula (4).

3.3 Mean-Variance framework

The variance-covariance framework is an approach to calculate Value-at-Risk taking into consideration correlations between assets. The simplest specification of this approach is based on historical variance-covariance estimation (Dowd, 2005). First, the window size n is chosen. Next, volatility is calculated by using variance-covariance matrix Σ_t estimated using last n observations. Then volatility σ_t is calculated by taking $\sqrt{w\Sigma_t w^T}$ where w is the vector of weights. Thereafter, under the assumption that portfolio's P/L distribution is $N(\mu, \sigma)$ one can calculate quantile by applying standard normal transformation (Alexander, 2008). Let μ_t be an arithmetic mean of portfolio returns. Let's set significance level $\alpha = 0.05$ and time horizon to $h = 1$. Then, portfolio's 95%VaR is expressed as:

$$VaR_t = -\Phi^{-1}\left(0.05\right)\sigma_t - \mu_t \tag{5}$$

where Φ^{-1} denotes the inverse of the CDF of the normally distributed random variable. If one assumes uniformly distributed weights for ten assets in the portfolio then (since Gaussian distribution is symmetrical), 95%VaR expressed as a percentage of the portfolio's value can be calculated as:

$$VaR_t = \Phi^{-1}\left(0.95\right)\sqrt{w\Sigma_t w^T} - \mu_t \approx 1.645\sqrt{0.01\sum_{i=1}^{10}\sum_{j=1}^{10} a_{i,j}} - \mu_t \tag{6}$$

where $a_{i,j}$ are coefficients of variance-covariance matrix Σ_t. Another possibility to evaluate VaR using variance-covariance matrix is to consider generalized t distribution instead of gaussian. Carol Alexander (2008) derives appropriate quantile with respect to standardized Student t distribution with mean 0 and variance 1.

$$VaR_{t,v} = -\sqrt{v^{-1}(v-2)}\, t^{-1}_{v,0.05}\sigma_t - \mu_t \tag{7}$$

Where σ_t and μ_t are defined as previously and $t^{-1}_{v,0.05}$ denotes the 0.05 quantile of the standardized Student t distribution with v the degrees of freedom. Preliminary data analysis showed that $v = 5$ might be a suitable choice for this parameter, which is subsequently discussed in detail. Considering symmetric properties of Student t distribution, and the form of the volatility estimator σ_t discussed above, 95%VaR formula follows equation:

$$VaR_{t,v} = \sqrt{v^{-1}(v-2)}\, t^{-1}_{v,0.95}\sqrt{w\Sigma_t w^T} - \mu_t \approx 2.015\sqrt{0.6}\sqrt{0.01\sum_{i=1}^{10}\sum_{j=1}^{10} a_{i,j}} - \mu_t \tag{8}$$

3.4 GARCH (1,1)

Model GARCH(p,q) created by Bollerslev (1986) is a generalization of the previously defined ARCH(p,q) model (Engle 1982). Since it's creation, ARCH-class models have been vastly used in modeling financial time series. Today, many extensions and modifications of these models are used in volatility modeling. A popular modification for GARCH modeling is changing assumption regarding the distribution of an error term (Zdanowicz, 2007; Szczerbak, 2017). In this paper two types of GARCH(1,1) models are discussed, the original one and one assuming Generalized Student t distribution of an error term.

3.4.1 GARCH(1,1) — Normal distribution

In it's simplest form, where $p = q = 1$, GARCH(1,1) is given by the set of equations:

$$z_t|\Psi_{t-1} \sim N(0, \sigma_t^2) \tag{9}$$

$$\sigma_t^2 = \omega + \alpha z_{t-1}^2 + \beta\sigma_{t-1}^2 \tag{10}$$

where z_t are IID random variables, Ψ_t is an information at the moment of t and α, β, ω are model's non-negative parameters. Under those specifications, we derive formulae for one-step-ahead volatility forecast. As previously, time window n is set to be 252 days. At any given moment t, volatility forecast for $t+1$ is calculated under the assumption that actual volatility at t–1 is a variance of the portfolio's return computed from the last 252 observations. Thus we get:

$$\sigma_{t+1}^2 = \omega_t + \alpha_t z_t^2 + \beta_t \sigma_{t\_empirical}^2 \tag{11}$$

Which corresponds to the equation in Engle (2001). After calculating series of daily volatilities, daily 95%VaR is then determined. In this scenario, the assumption of normally distributed portfolio returns is made. This means that VaR measurement can be calculated as in equation (5). This time, however, volatility estimator resulting from GARCH(1,1) specification is used. Hence, final formulae for calculating daily forecast for 95% VaR under the assumption of normally distributed returns is:

$$VaR_{t+1} = 1.645\sqrt{\sigma_{t+1}^2} - \mu_t \tag{12}$$

3.4.2 GARCH(1,1) — Generalized Student t distribution

An adjustment to GARCH(1,1) model can be made. In another article, Bollerslev (1987) introduced a modification of GARCH process. Rather unrealistic assumption about the normality of returns was waived and replaced by assuming the error term, as follows:

$$z_t|\Psi_{t-1} \sim T(v) \tag{13}$$

where $T(v)$ is Student distribution with v degrees of freedom, mean μ and variance h, which PDF is given by:

$$f(z_t|\Psi_{t-1}) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{\pi}\Gamma(\frac{v}{2})} \frac{[1 + \frac{z_t^2}{h_{t|t-1}(v-2)}]^{-\frac{v+1}{2}}}{\sqrt{(v-2)h_{t|t-1}}} \tag{14}$$

Equation (14) gives so-called generalized student t distribution. In this case, probability density function $p(z|v,\mu,\sigma)$, takes $\mu = 0$, $\sigma = \sqrt{\frac{h(v-2)}{v}}$ and $v = 5$. Because volatility shaping process is defined as in (10), we receive formula (11) for its forecasts, with notable difference that error term now follows different distribution. Having estimated volatility forecasts, one can proceed to compute 1-day 95%VaR according to equation (7), which can be rewritten as:

$$VaR_{t+1} = 2.015\sqrt{0.6\sigma_{t+1}^2} - \mu_t \tag{15}$$

3.5 EGARCH(1,1)

EGARCH(p,q) defined by Daniel Nelson (1991) is an extension of GARCH(p,q) model that aims to capture the asymmetry effect. Asymmetric reaction to the negative and positive news on the financial markets is a well-known fact. Exponential GARCH model addresses this problem. The need to capture the impact of information on volatility is crucial in it's modeling (Engle, 2002). In this subsection EGARCH(1,1) is presented. Studies have shown that distributions different than normal are eligible for modeling volatility (Harvey and Lange 2017). Therefore, to enrich this analysis, two specifications of EGARCH(1,1) are presented. EGARCH model in its simplest form is given by equations (16)-(18), where GED means Generalized Error Distribution, random variables z_t are IID and ω, β, α, and γ are the model's parameters.

$$Z_t \sim GED \tag{16}$$

$$log(\sigma_t^2) = \omega + g(z_{t-1}) + \beta log(\sigma_{t-1}^2) \tag{17}$$

$$g(z_t) = \alpha z_t + \gamma(|z_t| - E|z_t|) \tag{18}$$

3.5.1 EGARCH(1,1) — Normal distribution

In the first specification, it is assumed that variables z_t are normally distributed with mean 0 and variance 1. Since the expected value of $|z_t|$ is $\sqrt{\frac{2}{\pi}}$ (see Appendix 1), we get:

$$log(\sigma_t^2) = \omega + \alpha z_{t-1} + \gamma\left(|z_t| - \sqrt{\frac{2}{\pi}}\right) + \beta log(\sigma_{t-1}^2) \tag{19}$$

As previously, daily VaR is calculated via the one-step-ahead method, with an estimation window set to 252 days. Parameters are estimated daily and volatility forecast is given by:

$$\sigma_{t+1}^2 = e^{\omega_t + \alpha_t z_t + \gamma_t\left(|z_t| - \sqrt{\frac{2}{\pi}}\right)} (\sigma_{t\_empirical}^2)^{\beta_t} \tag{20}$$

Where $\sigma_{t\_empirical}$ is variance calculated from the estimation window, thus based on the past observations. Subsequently, substituting (20) into (12), daily 95%VaR is obtained.

3.5.2 EGARCH(1,1) — Generalized Student t distribution

Finally, let's specify the last analyzed model. Let's assume z_t are distributed according to distribution given by (14), where $v = 5$. In such a case, the expected value of random variable $|z_t|$ is equal to $\frac{4\sqrt{5}}{3\pi}$(see Appendix 1). Analogously to the equation (19) we get:

$$log(\sigma_t^2) = \omega + \alpha z_{t-1} + \gamma \left(|z_{t-1}| - \frac{4\sqrt{5}}{3\pi} \right) + \beta log(\sigma_{t-1}^2) \tag{21}$$

and consequently

$$\sigma_{t+1}^2 = e^{\omega_t + \alpha_t z_t + \gamma_t \left(|z_t| - \frac{4\sqrt{5}}{3\pi} \right)} (\sigma_{t\_empirical}^2)^{\beta_t} \tag{22}$$

Having estimated volatility time series, by substituting (22) into (12), we receive daily 95%VaR.

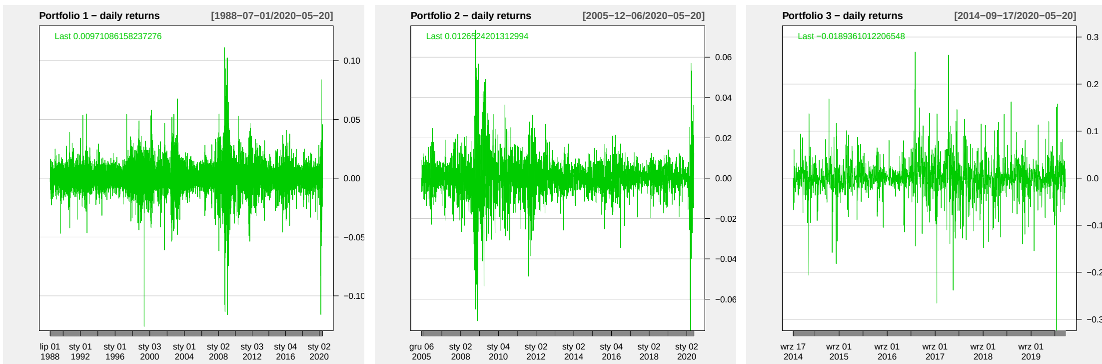
4 Data

Three sets of data have been used in the analysis. They were treated as three independent investments with different characteristics. Each volatility model has been applied to estimate Value at risk for each portfolio. In the study, historical prices of assets with daily frequency were used. In each case the daily price of an instrument is the adjusted close price, which takes into account change in value due to splits, dividends etc. All calculations were conducted using RStudio environment. In the selection process authors aimed to choose a small number of relatively long time series. This criterion was applied to properly estimate models and capture different macroeconomic events leading to periods of extremely high volatility. The character of this analysis enforces to ask oneself how accurately do volatility models forecast daily Value-at-Risk during a financial turmoil. Despite obvious insufficiency of VaR measurement a value of the study conducted in this paper could be the lesser, the shorter time period of the data was used. Thus, the analyzed period covers at least one major financial crisis. Another rule used in creating portfolios was to make sure that its constituents represent a reasonable amount of market capitalization, have desired level of liquidity, and in the case of the second set, is well diversified across the world and types of assets.

First portfolio is made of ten equal weights of the biggest firms from FTSE100 index. Namely; Unilever, British Petroleum, GlaxoSmithKline, Diageo, Rio Tinto, Prudential, Reckitt Benckiser, RELX, Royal Bank of Scotland and Barclays. All of the companies are present on the London Stock Exchange and make part of the Top 20 biggest companies (during the time of the study). Data were downloaded from Yahoo Finance database. Series starts on 01.07.1988 and ends on 20.05.2020, giving a total number of 8201 observations. This set of stocks represents hypothetical, well-diversified investment within one market of average size. Any risk manager could invest in a set of assets like this - list of stocks of firms with solid positions on the market, from different sectors, with a potentially low, but stable rate of return in long period of time. The second portfolio consists of various assets quoted on different markets around the world. As previously, weights are distributed equally amongst assets. Capital is invested in six indices (Bovespa, S&P 500, Nikkei 225, Euro Stoxx 50, KOSPI 200 and CRB Commodity Index), two currencies, one ETF and one REIT. Data are converted in US dollars (Bovespa, Nikkei 225, Euro Stoxx 50 and KOSPI 200 series are multiplied by exchange rates of BRL/USD, JPY/USD, EUR/USD and KRW/USD accordingly). Last two series of data represent alternative investment opportunities in non-traditional assets. One of them is Invesco Water Resources, a US exchange-traded fund (ETF), tracking the NASDAQ OMX US Water Index. It composes of US-listed firms involved with conservation and purification of water. Growing scarcity of this precious resource results in the growing popularity of investing in water-based industry's companies. Another one is Arbor Realty Trust which is Real Estate Investment Trust (REIT). Investing in real estates can be a good opportunity to enrich portfolio and Arbor's REIT is one of the biggest in the market. All of the data were downloaded from stooq website. First observation falls on 06.12.2005 and last on 20.05.2020. This gives a total number of 3761 observations. Third portfolio consists of four cryptocurrencies (Bitcoin, Litecoin, Ripple and Ethereum). Data were downloaded from Yahoo Finance database, all of the series are quoted in dolars. Three of them start on 17.09.2014 and end on 20.05.2020. Only in case of Ethereum, first observation falls on 07.08.2015. Therefore, the third portfolio was created as an equally weighted average of three assets until 07.08.2015 and then four assets after that date. Hence, lenghtening third portfolio's relatively short series to total numeber of 2073 observations.

One might observe that the first portfolio is quoted in pounds and the other ones in dollars. This does not influence this study because the calculation of volatility and VaR requires only the rate of return of assets. Since three sets of data are analyzed independently from each other, representing VaR in terms of percentage loss of a portfolio eliminates the problem of them being quoted in different currencies. It is worth mentioning that the goal of this study is not establishing the best set of assets for maximizing returns. Cumulative return from the investments is not crucial in the study, transaction costs are therefore not included. Preliminary analysis showed that individual observations were missing in the data. Missing values were completed with simple interpolation. Rate of returns of each portfolio are presented below.

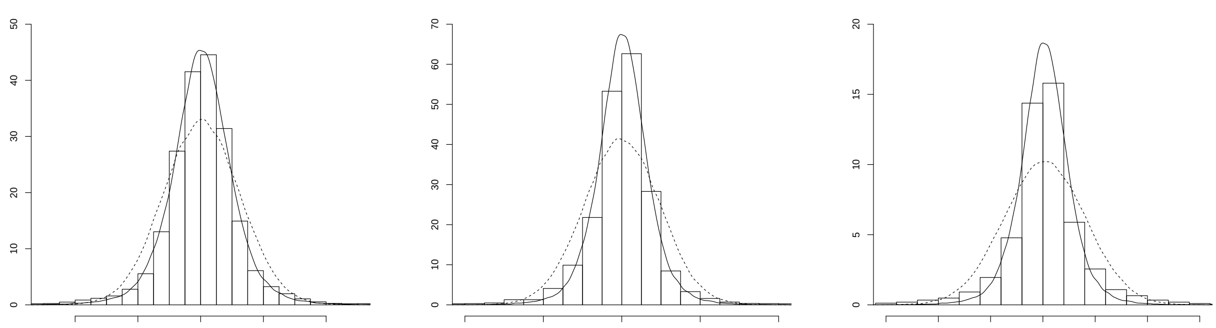
Figure 2: Portfolios' daily returns



Note: Chart on the left shows returns of the first portfolio, in the middle second and on the right the third one. Each series clearly show the volatility clustering effects. Second, more diversified portolio, shows lesser volatility, only slightly exceeding ±6%, unlike the first and third one showing over ±10% and ±20% accordingly.

Additionally, returns of their components are presented in Appendix 2 in Figures 9, 10 and 11. Volatility clustering is clearly visible in the presented data. After analyzing quantile plots of empirical distributions (see Figures 12, 13 and 14 in Appendix 3) it is safe to suspect, that these series have fat-tailed distributions. Histogram of returns has been generated. Dotted lines mark the density of Gaussian distribution, while continuous ones represent Student t with five degrees of freedom. Number of degrees of freedom was chosen in such a way, that distribution would fit the data best. Setting $v = 5$ is additionally beneficial because it is

Figure 3: Histogram's of portfolios' daily returns



Note: Chart on the left shows histogram of returns of the first portfolio, in the middle of the second portfolio and on the right, the third one. The continuous line is the density of student t distribution with 5 degrees of freedom and the dotted line is the standard normal density function.

the smallest number of degrees of freedom for which expected value, variance, skewness and kurtosis are determined. On the other hand with increasing v, t-distribution approaches Gaussian distribution. Therefore, the bigger the number of degrees of freedom, the more similar the results of the study will be. For these reasons setting $v = 5$ seems to be a reasonable assumption.

5 Empirical analysis

5.1 Hypotheses and research questions

Before VaR series were calculated, the couple of hypotheses have been considered regarding results. The main one is that GARCH(1,1) model with normally distributed returns is the most accurate of all of the analyzed models in terms of forecasting Value-at-Risk. Due to its popularity, it is worth considering the notion that the relatively simple model is the most useful one. Substantial level of complexity in building models is required from econometrician while simulating phenomenons on financial markets, however defining too sophisticated processes, covering too many effects may lead to data overfitting. Having that in mind and considering findings presented in many empirical studies (Poon and Granger, 2003) hypotheses for this research are as follow:

A. The best choice for forecasting Daily 95%VaR for a portfolio is GARCH(1,1) model with normally distributed returns assumption.

B. The mean-variance framework is significantly better for the second, diversified portfolio. Considering additional information by calculating the variance-covariance matrix helps if strong diversification between assets occurs.

C. Models using student t distribution are fitted better to the data, therefore in forecasting, those models tend to underestimate the risk due to the overfitting problem.

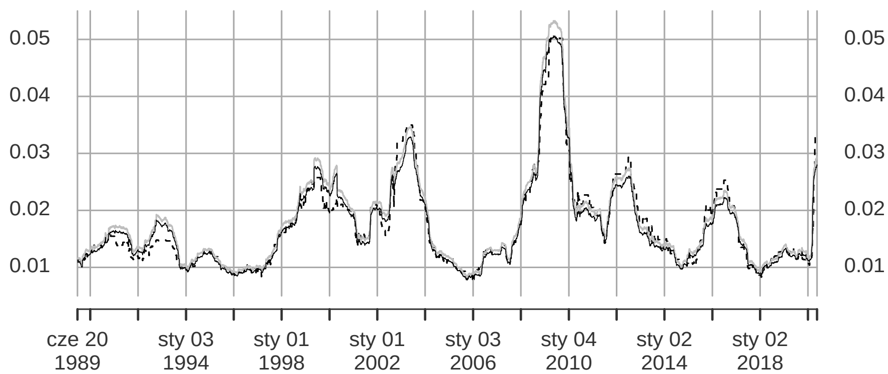
D. ARCH models are less conservative ways for estimating risk than basic methods.

E. Performance of the model is strongly dependent on the class of assets in the portfolio.

5.2 Calculation of Value-at-Risk

Calculations of VaR series were conducted using rolling estimation. For each model, the procedure looks alike. First parameters of the model were estimated on the 252 day in-sample period and then used to obtain the one-day-ahead volatility forecast. Next, VaR was calculated using the appropriate equation. Then, time window was moved by a day and procedure repeats.

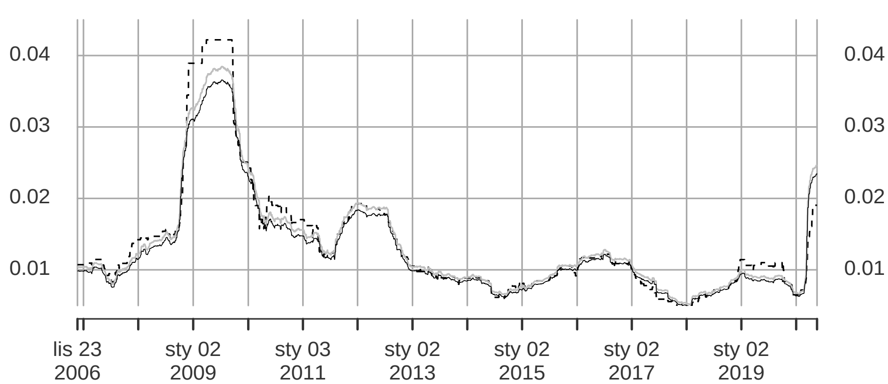
Figure 4: VaR series — parametric approaches. Portfolio 1.



Note: Historical Simulation is marked by a dotted line. Grey line represents parametric approach assuming normal distribution of returns and black, continuous line represents parametric approach assuming student t distribution.

Let's start presenting results of the study by showing together first three methods. On the figures below, VaR estimated using Historical Simulation is represented by a dotted line. Grey line shows VaR series obtained by parametric approach with normally distributed returns and continuous line with student t distribution with five degrees of freedom. Unsurprisingly, historical simulation seems to be the most conservative and the method using student t distribution the least. In case of a first Portfolio, until the beginning of 2002 highest estimates of

Figure 5: VaR series — parametric approaches. Portfolio 2.

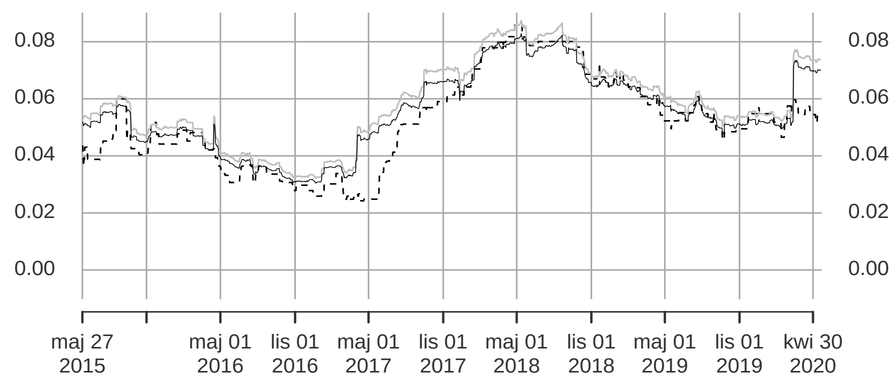


Note: VaR series are marked as in Figure 4.

VaR are obtained by the method using normal distribution assumption. After that, historical simulation shows bigger estimates. During higher volatility periods, the differences diverge.

In the case of the second Portfolio, discrepancies between methods are clearly smaller. It is not clear which of the method is less or more restrictive before further analysis. Similarities between them vanish during the 2008 financial crisis. During this time, historical simulation gives the highest estimations of risk, while parametric approach using student t distribution, the lowest.

Figure 6: VaR series — parametric approaches. Portfolio 3.

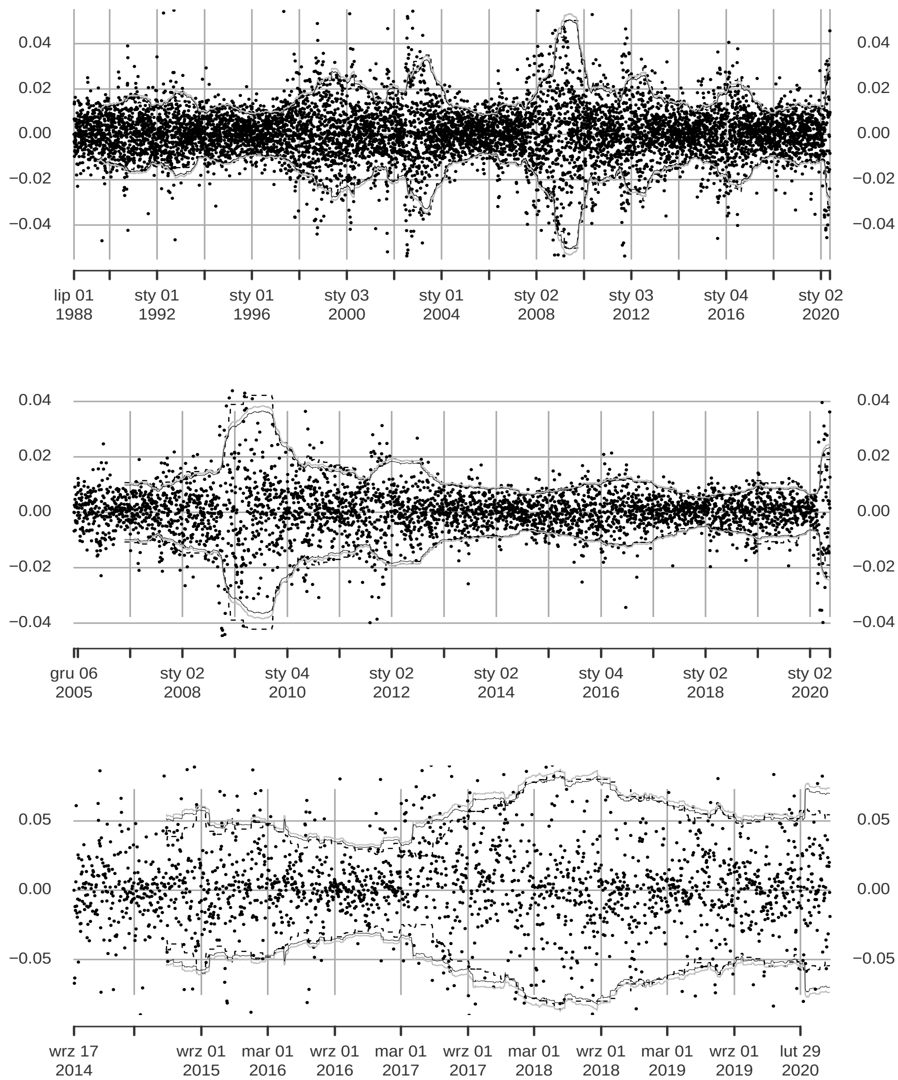


Note: VaR series are marked as in Figure 4.

The results for the third portfolio show visible differences in VaR levels. Methodology using student t distribution gives the highest estimate of VaR throughout the entire period. Historical simulation shows the lowest estimations, diverging significantly from the variance-covariance approaches. As observed before, during periods of high volatility, different parametric methods diverge from each other. Such volatility of returns is characteristic feature of cryptocurrencies, which migh be possible justification of this phenomenon. Especially big discrepancies occur near the end of the analyzed series. At one point in time, VaR obtained via historical simulation method deviates visibly. Due to extreme change in daily logarithmic returns for the third portfolio, values of the variance-covariance matrix increased permenently, shifting VaR series. Simultaniousily, this shock did not affect quantile obtained from empirical distibution. One might compare the differences by looking at portfolios' components rate of returns in Appendix 2.

For a clearer picture of the differences between estimations let's compare plots of portfolio's returns against parametric VaR series. Despite some previously noted observations, there is no clear pattern emerging as to which method is superior then other. During the times

Figure 7: Returns and parametric VaR estimation.

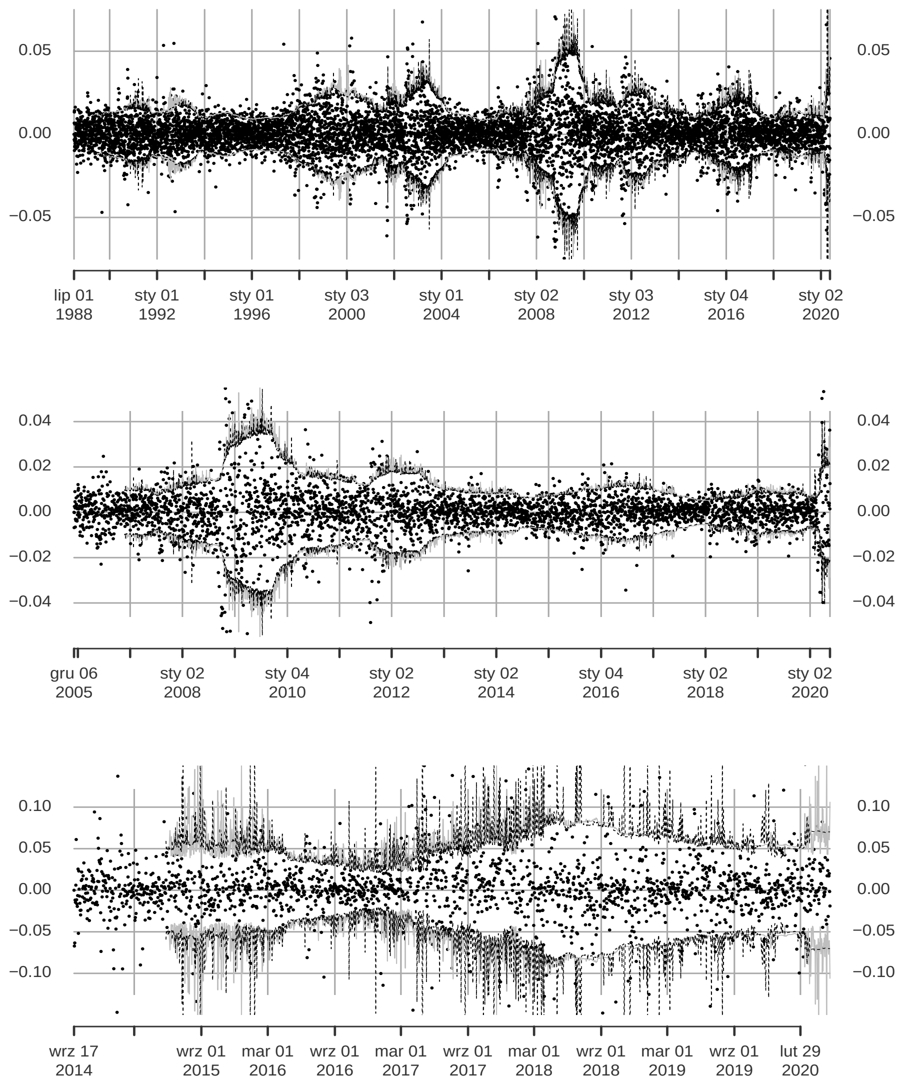


Note: VaR series are marked as in Figure 4. Dots represent logarithmic rate of returns of a portfolio. First portfolio is presented at the top, second one in the middle and the third one at the bottom.

of high volatility discrepancies between VaR estimations are intensifying. To examine this matter more carefully one might take a closer look at a behavior of the charts presented in the Figure 7 in the shorter and more volatile period of time.

In the case of the first Portfolio, historical simulation gives the lowest estimates of risk. It shows a strong resemblance to the series obtained by parametric approach with student t distribution. Surprisingly, in this case an approach using a normal distribution produces the highest VaR estimates, which are notably different than other two methods. As for the second

Figure 8: Returns and GARCH(1,1) VaR estimation.



Note: Grey line is the realization of VaR estimated with GARCH(1,1)-normal process and black, dotted line with GARCH(1,1)-std process. Dots represent logarithmic rate of returns of a portfolio. First portfolio is presented at the top, second one in the middle and the third one at the bottom.

Portfolio results are quite different. Parametric method with student t distribution gives the lowest estimates and historical simulation the highest. As in previous case series obtained by all methods are close to each other. Returns of the third Portfolio are definitely more dispersed than the other ones, which is a result of high volatility of cryptocurrencies series. Both parametric approaches give similiar results. Lowest estimates of VaR gives Historical Simulation approach. This is most visible at the end of analyzed period. ARCH models due to their stochastic nature look incredibly alike when presented on the chart. Therefore only GARCH(1,1) model is presented here. Realizations of GARCH(1,1) and EGARCH(1,1) processes for both distributions and each Portfolio are presented in appendix 4 in Figure 15. In Figure 8, VaR estimated using GARCH(1,1)-norm model is marked by grey continuous line and GARCH(1,1)-std by dotted, black line. Both realizations are relatively close to each other. Yet, they give significantly different results when subjected to backtesting procedures in the next chapter. To analyze the accuracy of VaR estimates, there is a necessity to present backtesting techniques subsequently.

5.3 Backtesting

The first step in assessing which model produces better forecasts is looking into number of exceedances. Table 1 shows summary of those statistics for all models in the study and for each Portfolio. To normalize those values, exceedances were divided by the number of expected exceedances. Given the fact that one day 95%VaR were estimated, the expected number of exceedances is $0.05 * 7949 \approx 397$ for the first Portfolio, $0.05 * 3509 \approx 175$ for the second Portfolio and $0.05 * 1821 \approx 91$ for the third. To evaluate A/E ratio in ARCH-models, Monte Carlo simulation was used. Each of the GARCH and EGARCH models were estimated 1500 times[5]. Then, out of those 1500 models, arithmetic mean of A/E ratios were derived. Those preliminary findings can be seen in Table 1. The interpretation of the A/E ratio is simple. If A/E > 1, then chosen method underestimates risk. It means that exceedances implied by our methodology occur more frequently than they should, given a 5% confidence level. As can be easily seen, for portfolio 1 and 2 every model potentially underestimates risks. For portfolio 3 however, only for Historical Simulation A/E ratio is greater and one. For the first Portfolio, GARCH(1,1)-norm, EGARCH(1,1)-norm and parametric approach with normally distributed returns turns out to be closest to one, therefore most resembling our expected outcome. In the next study, EGARCH(1,1)-norm and once again parametric approach with normally distributed returns turns out to be closest to 1. At the same time, this method is the most risk overestimating method for the third Portfolio. For cryptocurrencies methods using student t distribution of returns are closest to one. However, in case of first two portfolios models using student t distribution are (by this measure) much worse.

In order to determine whether or not this underestimation is statistically significant one ought to conduct a couple of more tests. Based on the premise that VaR exceedances follow Binomial distribution, Kupiec (1995) created statistical test (Binomial Kupiec test) that uses past

[5]Results of the study for 1500 estimations turned out to be accurate enough to draw meaningful conclusions. Estimating over 1500 realizations for each model would not bring any new information and would not change results in any way.

Table 1: Actual/Expected exceedances ratios.

	Portfolio 1		Portfolio 2		Portfolio 3	
	Exceedances	A/E exceedances ratio	Exceedances	A/E exceedances ratio	Exceedances	A/E exceedances ratio
Expected	397	–	175	–	91	–
HS	463	1.1662	214	1.2229	103	1.1319
Normal	418	1.0529	206	1.1771	79	0.8681
Student	471	1.1864	226	1.2914	87	0.9560
GARCH(1,1)-norm	429	1.0806	211	1.2057	81	0.8901
GARCH(1,1)-std	483	1.2166	230	1.3143	88	0.9670
EGARCH(1,1)-norm	430	1.0831	209	1.1943	81	0.8901
EGARCH(1,1)-std	488	1.2292	231	1.3200	90	0.9890

Note: Since each GARCH/EGARCH model is random process, exceedances for each portfolio was calculated by taking the arithmetic average of exceedances in 1500 VaR Series for that portfolio. This was motivated by the fact that after calculating the number of exceedances 1500 times, data sample started to resemble normal distribution and maximum likelihood estimator for mean in this distribution is given by its arithmetic average. (Appendix 5 shows histograms of all of the exceedances vectors.)

data to assess the validity of the model. In case of parametric methods, backtesting procedure is quite straightforward. Using the law of large numbers, binomial distribution is approximated by standard Gaussian distribution. Under the null that model is correct, p-value is determined. Following Lusztyn (2013) more than one test is calculated to determine the validity of the model. In the next step Christoffersen test (1998) is used to determine whether or not exceedances are independently distributed in time. Additionally mixed Christoffersen-Kupiec test is calculated to determine joint hypothesis that both null hypotheses are true. As it is observed in Table 2, there is no reason to reject the null only for four cases. Both Christoffersen and Mixed tests suggest rejecting null hypothesis for all models. Rejecting the null in Kupiec test means that model under or overestimates risk. Therefore, while estimating VaR of a Portfolio 1, one shall choose parametric normal approach. Simultaneously, in Portfolio 2 no method passed Kupiec Binomial test and in Portfolio 3, all of the methods passed it. Since the parametric apporach using student t distribution of returns showed better A/E ratio (closer to 1) it is potentially the best choice among parametric approaches to accurately calculate Value-at-Risk of a portfolio of cryptocurrencies.

Table 2: Backtesting — parametric methods. P-values of test statistics for both portfolios.

	Portfolio 1			Portfolio 2			Portfolio 3		
	Kupiec	Christoffersen	Mixed	Kupiec	Christoffersen	Mixed	Kupiec	Christoffersen	Mixed
HS	0.0010	5.2e-12	2.3e-14	0.0038	2.2e-11	3.4e-13	0.2078	6.2e-06	2.8e-06
Normal	0.2941	8.5e-10	4.9e-10	0.0211	3.8e-11	2.7e-12	0.1854	0.0037	0.0015
Student	0.0002	7.4e-13	8.9e-16	0.0002	7.2e-13	6.7e-16	0.6610	0.0032	0.0029

Note: Null hypothesis in Kupiec test is: the number of exceedances is correct, in Christoffersen null is: exceedances are independent from each other and in Mixed that both Kupiec and Christoffersen null hypotheses are true. Since both Kupiec and Christoffersen test statistics follow χ_1^2 distribution, then Mixed test statistic, being their sum, follows χ_2^2 distribution. Level of significance is 5 %.

The methodology used to obtain results from Table 2 can be transferred to ARCH-class models with a small adjustment. Using the Monte Carlo method, all four VaR series have been estimated 1500 times. Each time values of Kupiec, Christoffersen and Mixed tests were memorized, as well as the number of exceedances for each method. Those values were used in Table 3 to show fraction of those realizations that had p-values below 5%. As in the previous case, for first and second Portfolio, all models failed Christoffersen and Mixed tests. Hence in our data for all models the null hypothesis about the independence of exceedances have to be rejected. For third Portfolio in each case there is small percentage of cases where model passed Christoffersen or Mixed test but in general models tested for Portfolio 3 failed them as well. As for Kupiec test, the results are more varied. For Portfolio 1 methods using normal distribution passed the test most of the times. However all four models are unsuccessful for portfolio 2. Simultaneously, for portfolio 3 all models passed the Kupiec test. Looking back at Table 1 one might notice that GARCH(1,1)-norm and EGARCH(1,1)-norm have comparable A/E ratio for each Portfolio. That would suggest that GARCH(1,1)-norm and EGARCH(1,1)-norm are two most accurate models when forecasting VaR using apporach based on analyzed ARCH class models.

Table 3: Backtesting — ARCH models. Percentage of cases where p-values of test statistics were less then 0.05.

	Portfolio 1			Portfolio 2			Portfolio 3		
	Kupiec	Christoffersen	Mixed	Kupiec	Christoffersen	Mixed	Kupiec	Christoffersen	Mixed
GARCH(1,1)-norm	5.7 %	100 %	100 %	99.9 %	100 %	100 %	0.5 %	98.9 %	97.2 %
GARCH(1,1)-std	100 %	100 %	100 %	100 %	100 %	100 %	0 %	99.3 %	97.3 %
EGARCH(1,1)-norm	19.1 %	100 %	100 %	98.1 %	100 %	100 %	0.6 %	99.8 %	99.3 %
EGARCH(1,1)-std	100 %	100 %	100 %	100 %	100 %	100 %	0 %	98.7 %	94.7 %

Note: Null hypothesis in Kupiec, Christoffersen and Mixed tests are the same as in Table 2. Each value in Table 3 represents percentage of 1500 times in the Monte Carlo simulation when p-values of test statistic for process realization were less then 0.05.

5.4 Verification of hypotheses

To summarize, given the results obtained in Table 2 and Table 3, the best choice for one day 95%VaR estimation for Portfolio 1 and 2 would be mean-variance framework (from parametric family) and GARCH(1,1) method (from non-parametric), both assuming normally distributed rate of returns. For Portfolio 3, results in Table 1 show that the parametric and non-parametric approaches with the student t distribution have a slight advantage over the other. Additionally, there is no sufficient evidence in Table 2 to confirm the main hypothesis of this thesis, that the best choice for forecasting Daily 95%VaR for a portfolio is GARCH(1,1)-norm model. Despite, the GARCH(1,1)-normal model appears to be the best choice for estimating Daily 95%VaR of a portfolio within the family of analyzed ARCH-type models, there is no observable advantage in using GARCH(1,1)-normal model over the mean-variance framework. Therefore, the hypothesis A can be rejected based on the results of this study. Nevertheless, the mean-variance framework's results seem not to be sensitive to the change in a portfolio's diverisification level. Thus, the hypothesis B about effects of diversification on estimating Value-at-Risk did not found justification nor confirmation. That being said, one could argue that for Portfolio 3 parametric methods appear to give observably better results. Moreover, despite some significant differences between two portfolios, there is no clear pattern emerging from the findings that the performance of a model is strongly connected to its constituents. Therefore, hypothesis E can be rejected. As for the other hypotheses, hypothesis C found its justification in the results for the first two portfolios. Underestimation of the risk by models using student t distribution turned out to be true. None of the analyzed models improved it's forecasting accuracy by adopting this distribution. As for the third portfolio, every model using this distribution overestimates risk, however not as much as models using the normal distribution of returns. Additionally, analysis of the exceedances in Table 1 shows, that in general, GARCH and EGARCH models do not give lower estimates of risks then parametric models. Hence, hypothesis D is rejected, as well. ARCH models might not be less conservative than basic methods.

6 Summary and conclusions

In the study, seven different approaches to model Value-at-Risk are analyzed. Starting with parametric methods: Historical Simulation, Mean-Variance framework (assuming normal and student t distribution of returns), through non-parametric ARCH-class models: GARCH(1,1) and EGARCH(1,1) (both assuming normal and student t distribution of returns).

Three sets of data were taken into consideration. Daily prices were used to create equally weighted portfolios of ten different assets. In the case of the first portfolio, stocks of the biggest firms from FTSE100 index were used. All of the data were downloaded form the Yahoo Finance database. Series starts on 01.07.1988 and ends on 20.05.2020, giving combined 8201 observations. In the case of the second portfolio, different kinds of assets were used. It consists of six indices, two currencies, one ETF and one REIT. Data were dowloaded from stooq website. Series begins on 06.12.2005 and end on 20.05.2020, giving a total number of 3761 observations. Third data set consists of four cryptocurrencies. Data downloaded from Yahoo Finance begins on 17.09.2014 and ends on 20.05.2020 giving total number of 2073 observations.

The results of the study conducted on three different portfolios do not show clear dominance of the GARCH(1,1)-normal model in estimating daily 95%Value-at-Risk. If one were to consider parametric approaches, the mean-variance framework assuming normally distributed returns would be the best choice. It is worth mentioning here though, that in case of a very large portfolio, meanvariance framework is significantly more time-consuming than GARCH(1,1)-normal approach. Therefore, for sufficiently large set of assets, the latter model could be superior from the former one despite its complexity. Hypotheses concerning the lack of predictive power of models constructed using student t distribution were confirmed for at least first two portfolios. Overall, analyzed ARCH models turned out not to be less conservative ways to measure risk than basic methods. There is no reason to assume that performance of the model depends on the type of assets in the portfolio. Additionally, hypothesis that the mean-variance framework gives more accurate forecasts for more diversified portfolio did not found confirmation.

Although the results obtained in this study are not definitive, this paper shows that building complicated models, covering various economic effects not always leads to better forecasting power. That said, this study does not exhaust the vast subject of VaR estimation. All models defined above can be calculated using a different set of data and assuming another level of confidence. Further research could therefore concentrate on studying the sensitivity of VaR changes depending on holding period and confidence level. Furthermore, only the most popular specifications of the aforementioned models were analyzed. Exploring the criteria of choice of the p,q parameters in GARCH(p,q)/EGARCH(p,q) models could also be beneficial for estimating Value-at-Risk.

Appendix 1

Derivation of expected value used in EGARCH(1,1)-normal model:

Let $X_t \sim N(0,1)$, then

$$E(|X|) = \int_{\mathbb{R}} |x|\,\varphi(x)\mathrm{d}x = -\int_{-\infty}^{0} x\,\varphi(x)\mathrm{d}x + \int_{0}^{+\infty} x\,\varphi(x)\mathrm{d}x = 2\int_{0}^{+\infty} x\,\varphi(x)\mathrm{d}x =$$
$$= 2\int_{0}^{+\infty} \frac{x}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_{0}^{+\infty} x\,e^{-\frac{x^2}{2}}\mathrm{d}x = \sqrt{\frac{2}{\pi}}$$

Derivation of expected value used in EGARCH(1,1)-student model:

Let $X_t \sim T(5)$, then

$$E(|X|) = \int_{\mathbb{R}} |x|\,f(x)\mathrm{d}x = -\int_{-\infty}^{0} x\,f(x)\mathrm{d}x + \int_{0}^{+\infty} x\,f(x)\mathrm{d}x = 2\int_{0}^{+\infty} x\,f(x)\mathrm{d}x =$$
$$= 2\int_{0}^{+\infty} x\frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\,\Gamma(\frac{v}{2})}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}\mathrm{d}x = \frac{2\,\Gamma(3)}{\sqrt{5\pi}\,\Gamma(\frac{5}{2})}\int_{0}^{+\infty} x\left(1+\frac{x^2}{5}\right)^{-3}\mathrm{d}x =$$
$$= \frac{16}{3\sqrt{5}\pi}\int_{0}^{+\infty}\frac{125x}{(5+x^2)^3}\mathrm{d}x = \frac{2000}{3\sqrt{5}\pi}\left(\lim_{x\to+\infty}\frac{-1}{4\,(5+x^2)^2} + \lim_{x\to 0^+}\frac{1}{4\,(5+x^2)^2}\right) =$$
$$= \frac{4\sqrt{5}}{3\pi}$$

Appendix 2

Figure 9: Logarithmic rate of returns of assets from portfolio 1.

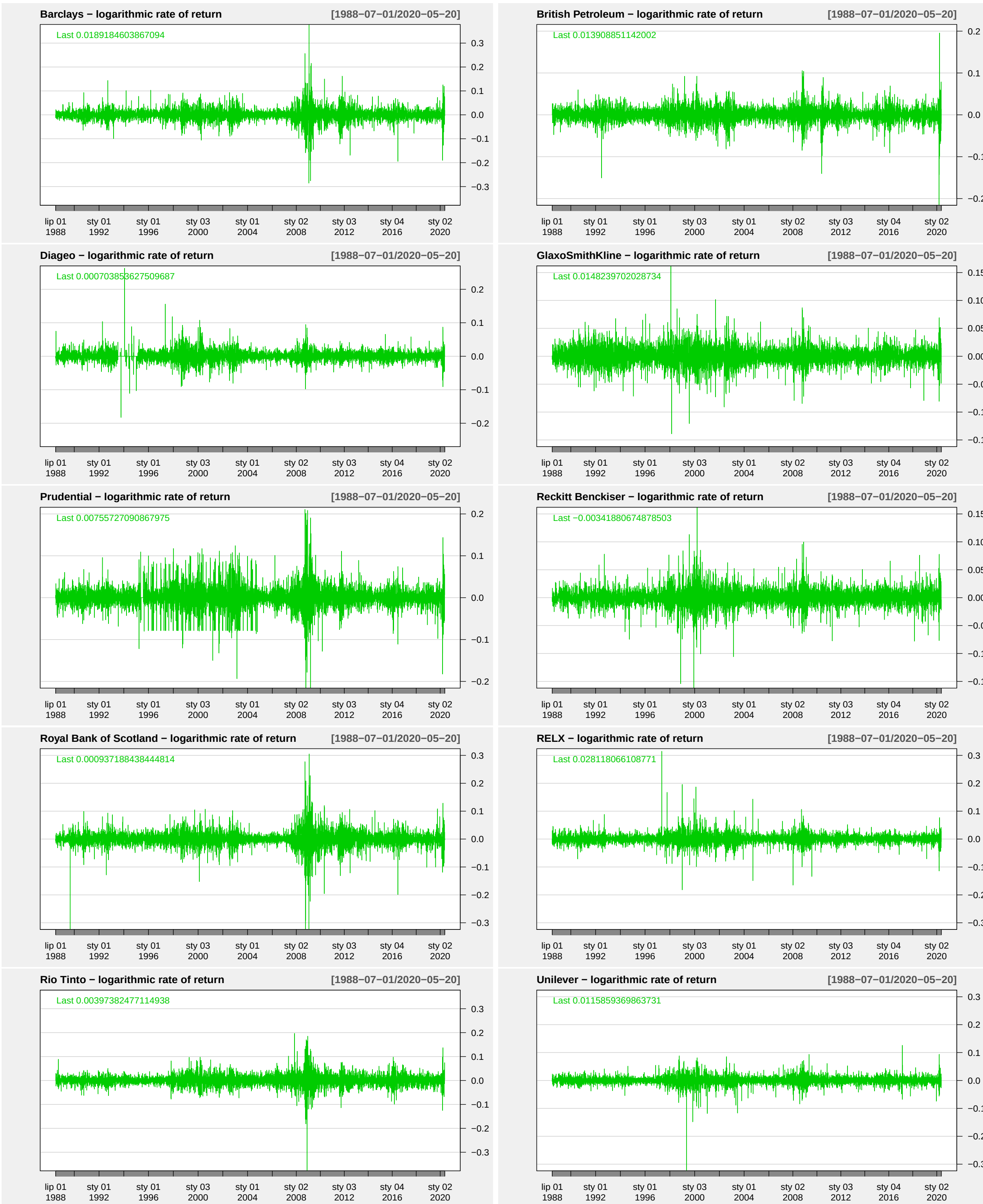


Figure 10: Logarithmic rate of returns of assets from portfolio 2.

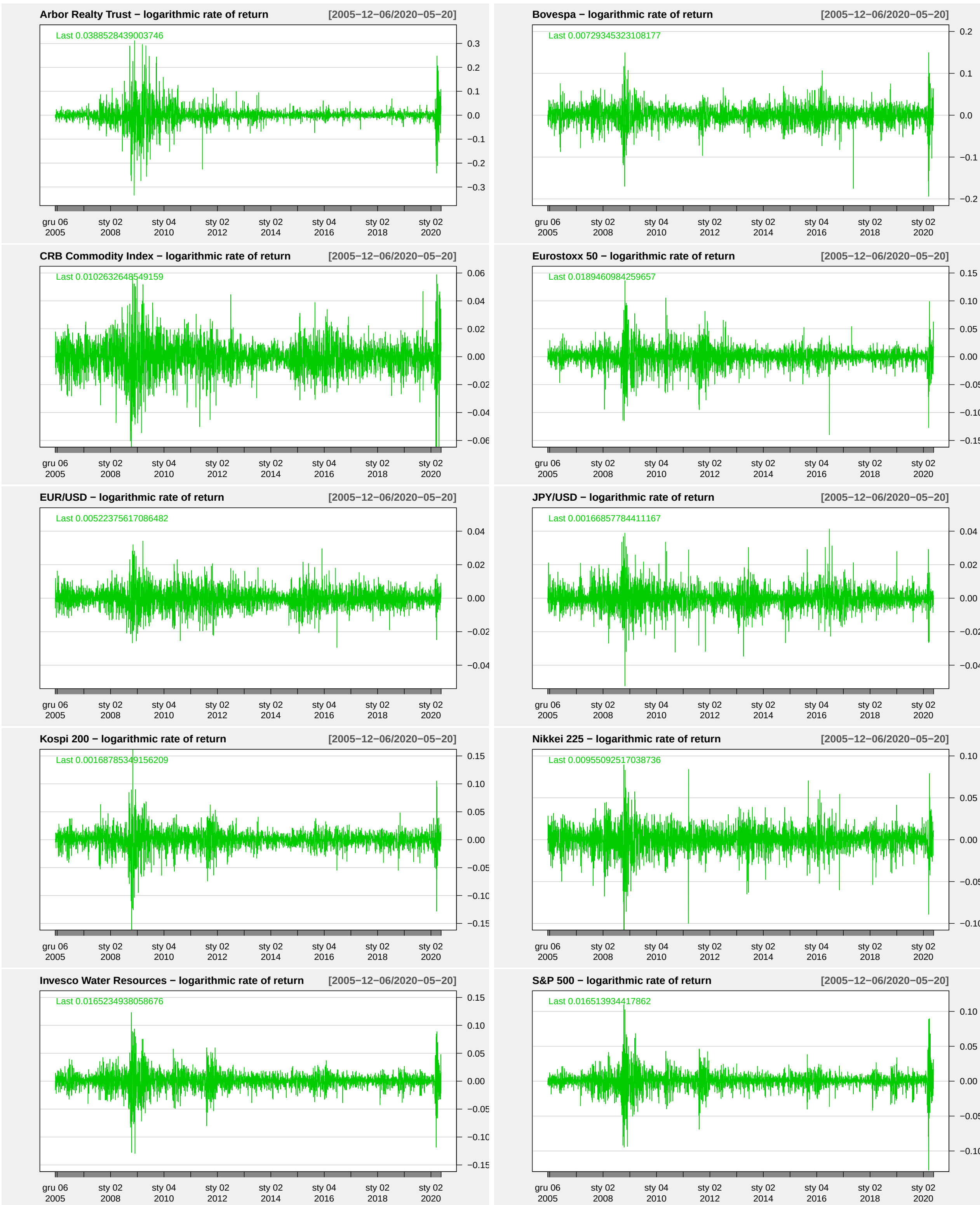
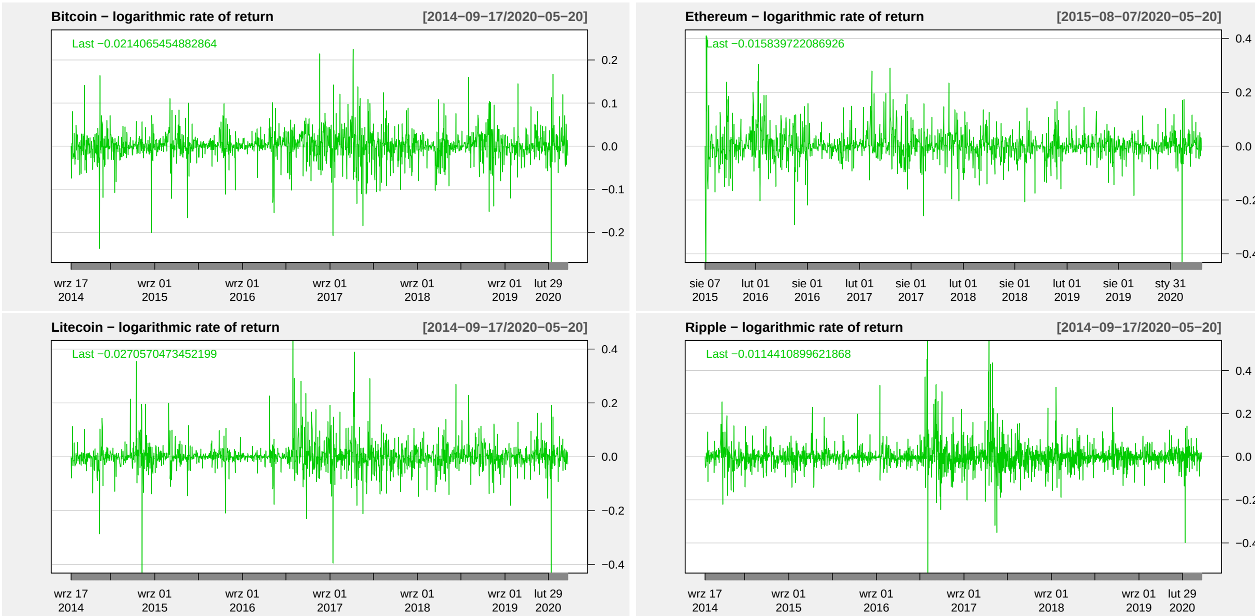
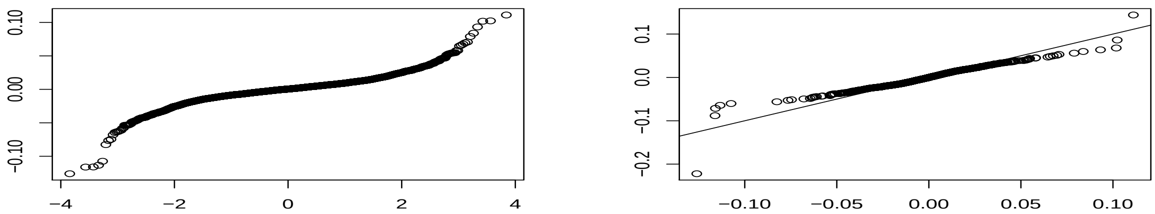


Figure 11: Logarithmic rate of returns of assets from portfolio 3.



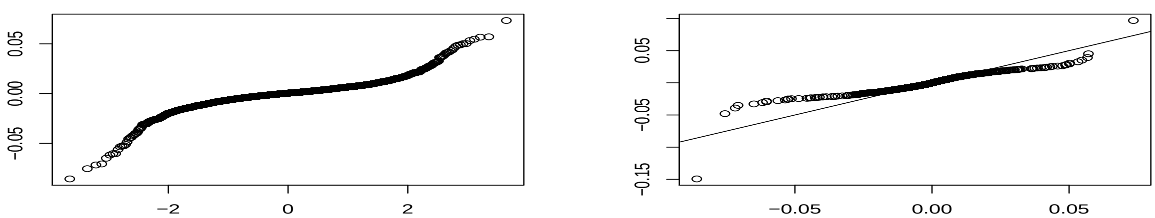
Appendix 3

Figure 12: Q-Q plots for portfolio 1.



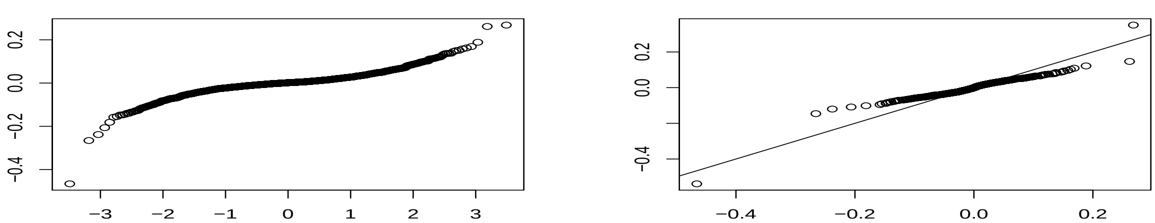
Note: On the left plot quantiles from empirical distribution are presented on the horizontal axis and quantiles from normal distibution on the vertical. On the right plot vertical axis represents quantiles from student t distribution.

Figure 13: Q-Q plots for portfolio 2.



Notion: Figure 13 shows analogous charts to Figure 12, however for Portfolio 2.

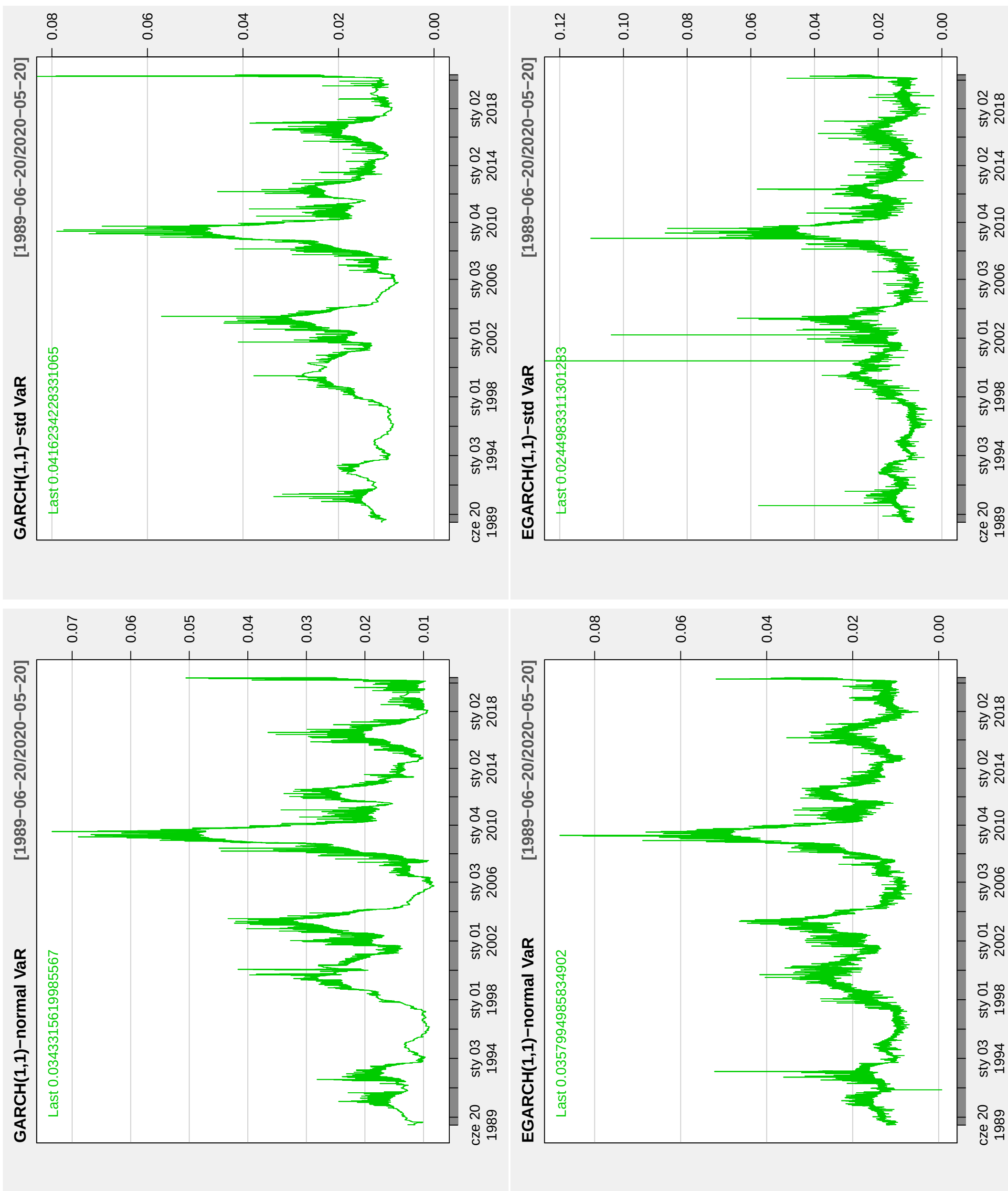
Figure 14: Q-Q plots for portfolio 3.



Notion: Figure 14 shows analogous charts to Figure 12, however for Portfolio 3.

Appendix 4

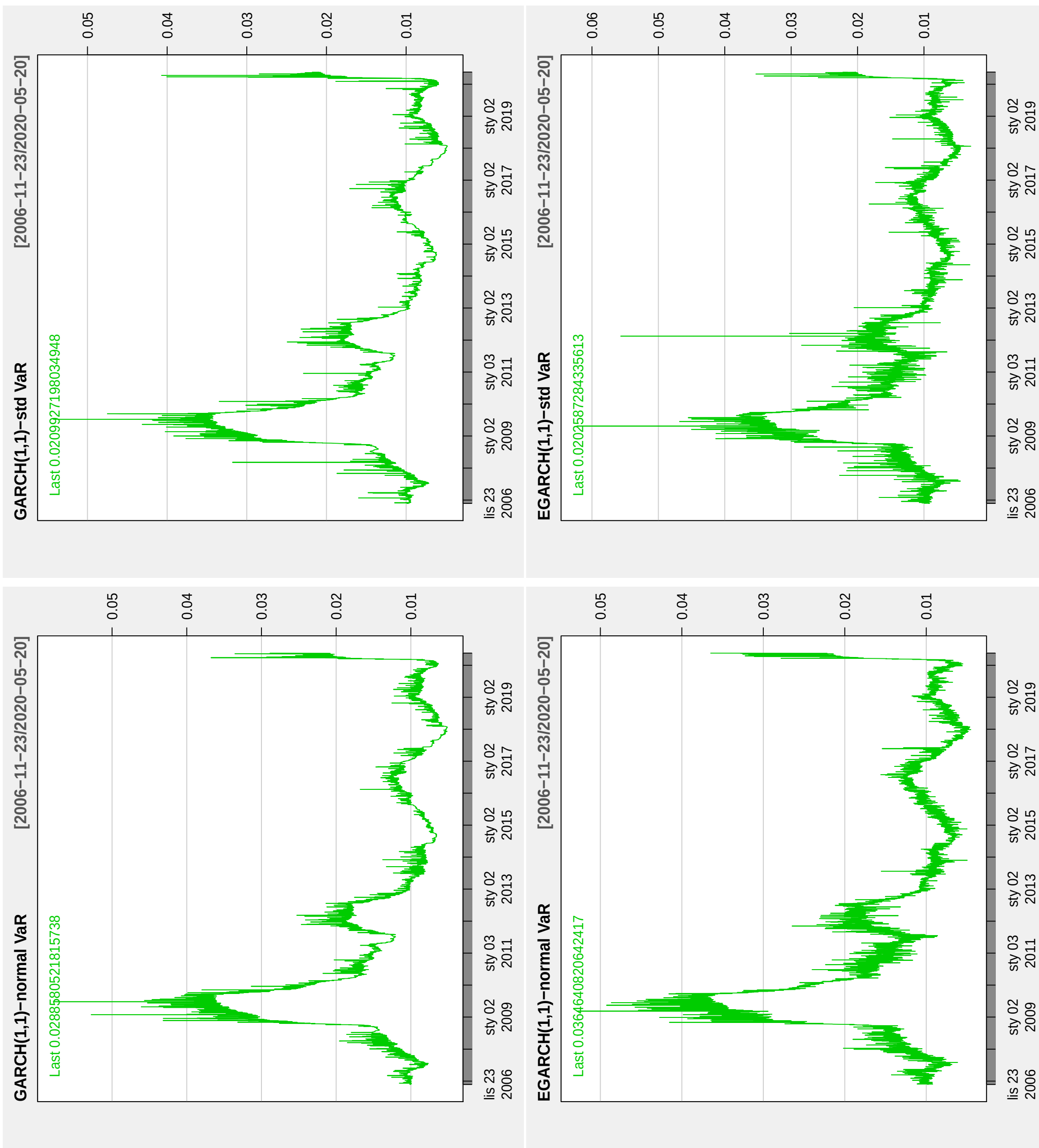
Figure 15: Value-at-Risk realizations for ARCH-class models. Portfolio 1.



Note: Figure 15 shows 1 of 1500 analyzed VaR series estimated using different volatility forecasts. On the upper left there is GARCH(1,1)-normal approach and below, GARCH(1,1)-student *t*. On the upper right EGARCH(1,1)-normal approach is represented, while below, EGARCH(1,1)-student *t* estimates follows.

Appendix 4, continued

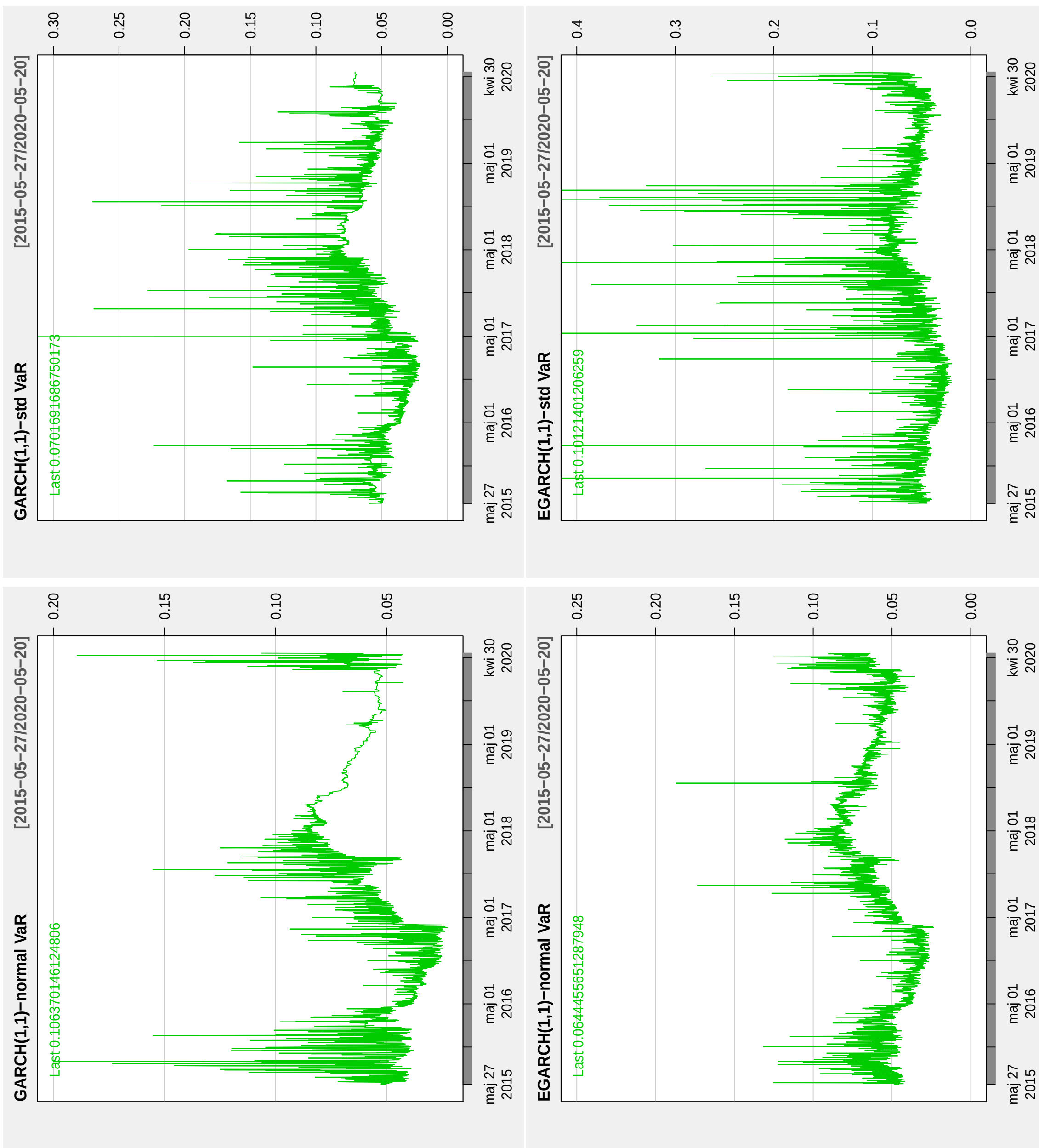
Figure 16: Value-at-Risk realizations for ARCH-class models. Portfolio 2.



Note: Figure 16 analogous charts to 15, however for Portfolio 2.

Appendix 4, continued

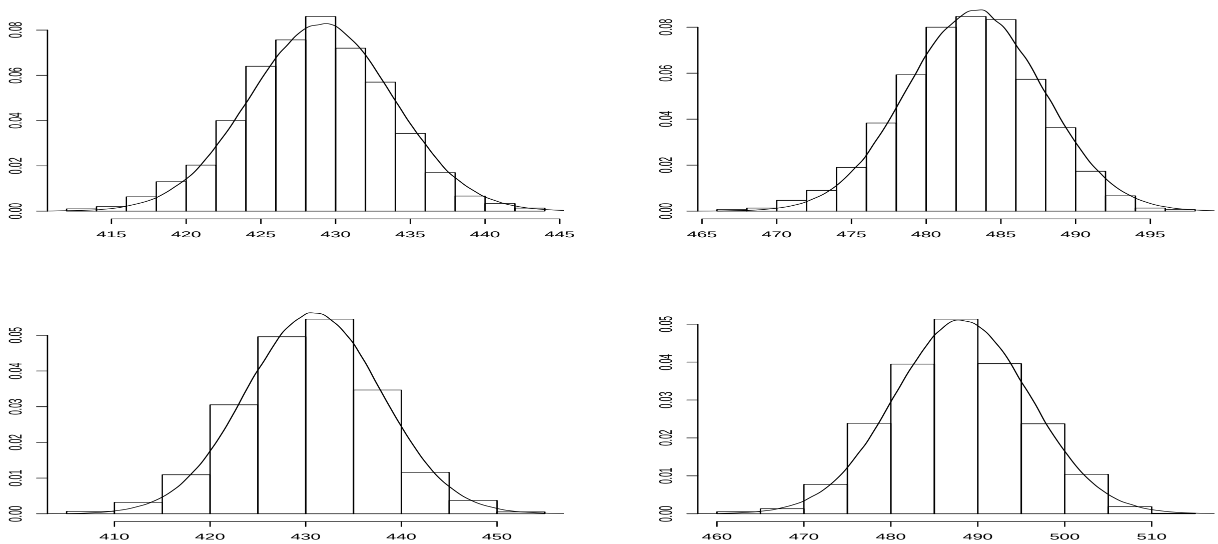
Figure 17: Value-at-Risk realizations for ARCH-class models. Portfolio 3.



Note: Figure 17 analogous charts to 15, however for Portfolio 3.

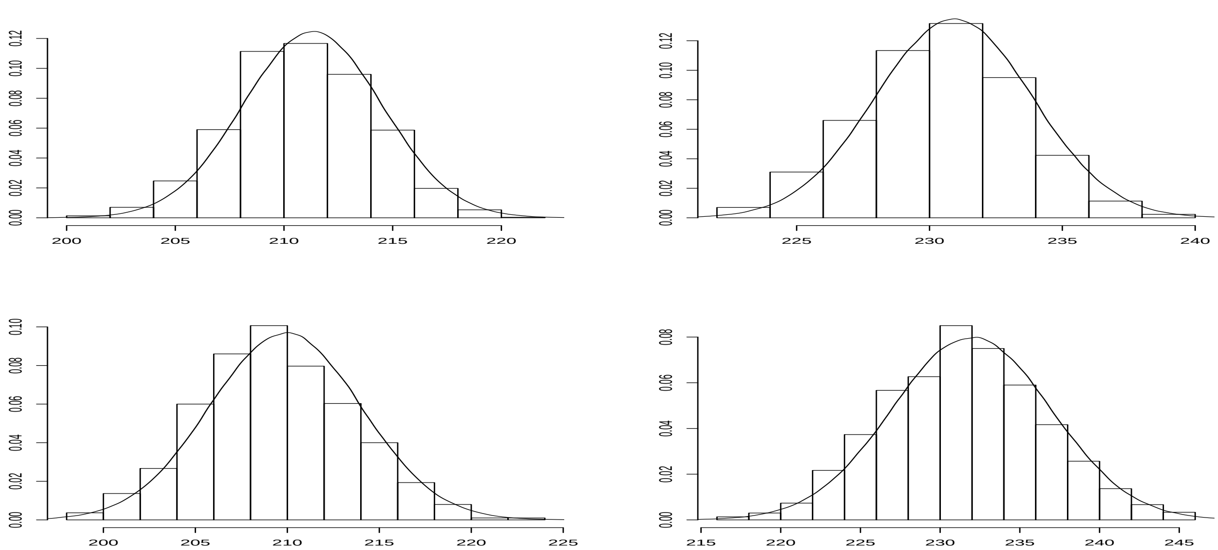
Appendix 5

Figure 18: Histograms of vectors of exceedances. Portfolio 1.



Note: On the upper left histogram of exceedances for GARCH(1,1)-norm is presented. On the right histogram of GARCH(1,1)-std is placed. Below there is histogram of exceedances for EGARCH(1,1)-norm on the left and EGARCH(1,1)-std on the right.

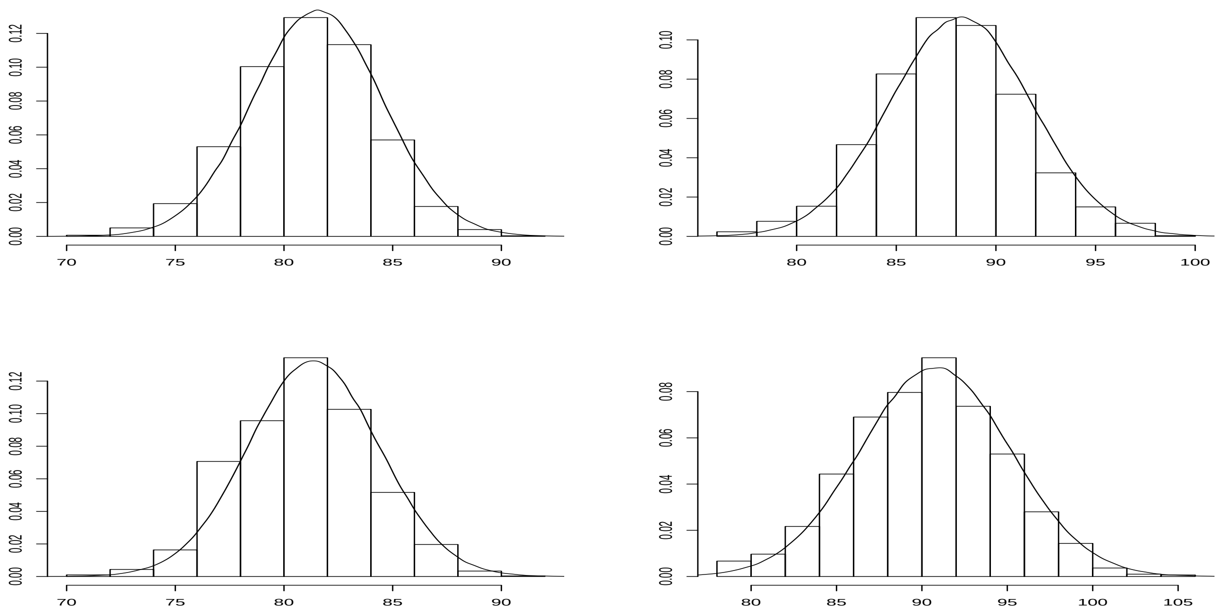
Figure 19: Histograms of vectors of exceedances. Portfolio 2.



Note: Histograms are sorted the same way as in Figure 18.

Appendix 5, continued

Figure 20: Histograms of vectors of exceedances. Portfolio 3.



Note: Histograms are sorted the same way as in Figure 18.

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