Course Handouts:

INTRODUCTION TO ASSET PRICING THEORY

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MOKD, Warsaw, December 2010

1. Security Markets and Portfolio Choice

Security Markets. (1.2)

Two time periods (dates): $t = 0$ is present; $t = 1$ is future.

Uncertainty at the future date 1 is described by a set $S = \{1, \ldots, S\}$ of states of nature.

J securities. The payoff of security j at date 1 is $x_j = (x_{j1}, \ldots, x_{jS})$.

Examples of securities: Stocks (i.e., equity shares), bonds, and many others. For a risk-free bond with face value 100 the payoff is state-independent with $x_{js} = 100$ for every s.

$J \times S$ matrix X with rows x_j is the **payoff matrix**.

A portfolio is denoted by $h = (h_1, \ldots, h_J)$. Its payoff in state s is $\sum_j x_{js} h_j$.

Asset span $\mathcal{M}$ is the set of payoffs of all portfolios. Formally,

$$\mathcal{M} = \{z \in I\!R^S : z_s = \sum_j x_{js} h_j, \quad \forall s, \text{ for some portfolio } h\}.$$

If $\mathcal{M} = I\!R^S$, then markets are **complete**; otherwise markets are **incomplete**.

For markets to be complete it is necessary and sufficient that there exist S securities with linearly independent payoffs, or, in other words, that the payoff matrix X have rank S. Of course, this requires that $J \geq S$.

A security is **redundant** if its payoff can be obtained as the payoff of a portfolio of other securities.

The price of security j at date 0 is p_j.

The **return** of security j is

$$r_j = \frac{x_j}{p_j}.$$

If there is a risk-free security, then its risk-free return is denoted by $\bar{r}$.

Agents. (1.3)

Consumption (of a single consumption good) takes place at date 0 and at date 1. Date-1 consumption plan specifies consumption conditional on each state, that is, it is state contingent. Typical consumption plan for dates 0 and 1 is $(c_0, c_{11}, \ldots, c_{1S}) \in \mathcal{R}_+^{S+1}$.

Agent's preferences are described by utility

$$u(c_0, c_1)$$

with **continuous utility function** $u : \mathcal{R}_+^{S+1} \to \mathcal{R}$.

Endowments are $w_0 \in \mathcal{R}_+$ at date 0 and $w_1 \in \mathcal{R}_+^S$ at date-1.

Consumption-Portfolio Choice. (1.4)

$$\max_{c_0, c_1, h} u(c_0, c_1) \tag{1.4}$$

subject to

$$c_0 + ph \leq w_0, \qquad c_{1s} \leq w_{1s} + \sum_j x_{js} h_j, \ \forall s$$

and $c_0 \geq 0, \ c_1 \geq 0$

First-Order Conditions. (1.5)

If function u is differentiable, the first-order conditions for a solution to the consumption and portfolio choice problem imply (assuming that the solution is interior, and that $\partial_0 u > 0$) that

$$p_j = \sum_s x_{js} \frac{\partial_s u}{\partial_0 u} \qquad \forall j. \tag{1.14}$$

The price of security j is equal to the sum over states of its payoff in each state multiplied by the marginal rate of substitution between consumption in that state and consumption at date 0. In other words, price equals marginal value of payoff.

Notes: This part was based on Chapters 1 and 2 from LeRoy and Werner (2001).

Arbitrage at prices p is a portfolio $\hat{h}$ such that

$$x_s \hat{h} \geq 0 \ \text{ for every } s, \ \text{ and } p\hat{h} \leq 0,$$

with at least one strict inequality. That is, a portfolio with positive payoff and negative price, with either the payoff or the price different from zero.

Strong arbitrage is a portfolio $\hat{h}$ with $p\hat{h} < 0$ and $x_s \hat{h} \geq 0$ for every s. That is, positive payoff and strictly negative price.

Security price vector p is **arbitrage free** if there is no arbitrage at p.

Arbitrage and Optimal Portfolios. (3.6)

Theorem 3.6.1: *If at given security prices an agent's optimal portfolio exists, and if the agent's utility function is strictly increasing, then there is no arbitrage.*

The absence of arbitrage is also a sufficient condition for the existence of an optimal portfolio.

Theorem 3.6.5: *If at given security prices there is no arbitrage, and if the agent's consumption is restricted to be positive, then there exists an optimal portfolio.*

State Prices and the Absence of Arbitrage.

Theorem (5.2.1 and 6.2.1): *Security price vector p is arbitrage free if and only if there exist strictly positive numbers $\{q_s\}_{s=1}^{S}$ such that*

$$p_j = \sum_{s=1}^{S} q_s x_{js} \quad \forall\ j. \tag{6.5}$$

Proof: This follows from Stiemke's Lemma 6.3.1, see Appendix. The proof in the books follows from Theorems 5.2.1 and 6.2.1.

Strictly positive numbers $\{q_s\}$ satisfying (6.5) are called **state prices.**

Let p be an arbitrage-free security price vector. If markets are complete, then there is a unique vector of (strictly positive) state prices. If markets are incomplete, then there are multiple vectors of state prices. This is so because (6.5) is a system of J equations with S unknowns q_s. It has a unique solution iff markets are complete.

Example: Let there be two securities with payoffs $x_1 = (1, 1, 1)$ and $x_2 = (0, 1, 2)$. Let $p_1 = 0.8$, and $p_2 = 1.2$. The two equations for state prices are

$$q_1 + q_2 + q_3 = 0.8, \qquad q_2 + 2q_3 = 1.2.$$

There exist strictly positive state prices. For example $(0.1, 0.2, 0.5)$. So p is arbitrage free. General solution for state prices is

$$q_1 = q_3 - 0.4, \quad q_2 = 1.2 - 2q_3, \quad \text{with} \quad 0.4 < q_3 < 0.6.$$

Risk-Neutral Probabilities. (6.7)

Suppose that there is a risk-free security with return $\bar{r} > 0$. Let q be a strictly positive state-price vector. Define

$$\pi_s^* = \bar{r} q_s,$$

or, equivalently

$$\pi_s^* = \frac{q_s}{\sum_s q_s}.$$

So defined, the π_s^*'s are strictly positive probabilities. We call them **risk-neutral probabilities**.

Using risk-neutral probabilities, we can rewrite equation (6.5) as

$$p_j = \frac{1}{\bar{r}} \sum_{s=1}^{S} \pi_s^* x_{js} \quad \forall j,$$

that is

$$p_j = \frac{1}{\bar{r}} E^*(x_j) \quad \forall j, \tag{6.22}$$

where E^* is the expectation under π^*. Thus, the price of security j equals the expectation of the payoff under π^* discounted by $\bar{r}$.

Risk-neutral probabilities have all the properties that state prices have.

Payoff Pricing Functional

We define the **payoff pricing functional** $q : \mathcal{M} \to \mathcal{R}$ as

$$q(z) = ph \ \text{ for some } h \text{ such that } z = hX.$$

Function q assigns to each payoff in the asset span $\mathcal{M}$ the price of the portfolio that generates that payoff. q is a linear functional on $\mathcal{M}$.

If there is no arbitrage, then, for any payoff z in the asset span $\mathcal{M}$, we have

$$q(z) = \sum_{s=1}^{S} q_s z_s.$$

Thus, the price of z can be calculated (uniquely) using state prices.

If state claim $e_s = (0, \dots, 1, \dots, 0)$ for state s lies in $\mathcal{M}$, then its price is equal to q_s. This is why q_s is called state price.

Also

$$q(z) = \frac{1}{\bar{r}} E^*(z) \quad \forall z \in \mathcal{M}.$$

Valuation of Contingent Claims

Valuation functional is a strictly positive extension of the pricing functional to the entire contingent claim space $\mathcal{R}^S$. Formally, valuation functional is $Q : \mathcal{R}^S \to \mathcal{R}$ such that

(i) Q coincides with the payoff pricing functional on $\mathcal{M}$, that is

$$Q(z) = q(z) \quad \text{for every } z \in \mathcal{M},$$

(ii) Q is strictly positive.

The Fundamental Theorem of Finance, 5.2.1: *Security prices exclude arbitrage iff there exists a strictly positive valuation functional.*

Every strictly positive vector of state prices $\{q_s\}$ defines a valuation functional

$$Q(z) = \sum_{s=1}^{S} q_s z_s, \tag{6.3}$$

for every $z \in \mathcal{R}^S$.

Valuation functional Q can be written using risk-neutral probabilities as

$$Q(z) = \frac{1}{\bar{r}} E^*(z). \tag{6.24}$$

If markets are incomplete and security prices are arbitrage free, then there are multiple vectors of state prices. Each vector gives rise to a valuation functional; hence, there are multiple valuation functionals. If markets are complete, then the unique valuation functional is the payoff pricing functional.

Value Bounds. (5.3, 6.5)

For any contingent claim $z \in \mathcal{R}^S$, we define the upper and the lower bounds on the value of z by

$$q_u(z) \equiv \min_h \{ph : hX \geq z\}, \tag{5.3}$$

$$q_\ell(z) \equiv \max_h \{ph : hX \leq z\}. \tag{5.4}$$

If there is no arbitrage, then

$$q_u(z) = q_\ell(z) = q(z)$$

for every $z \in \mathcal{M}$, see Proposition 5.3.1. Also

$$q_\ell(z) < \sum_{s=1}^{S} q_s z_s < q_u(z)$$

for every $z \notin \mathcal{M}$ and for every strictly positive state-price vector q, see Proposition 5.3.5.

It holds

$$q_u(z) = \sup_q \sum_{s=1}^{S} q_s z_s, \qquad q_\ell(z) = \inf_q \sum_{s=1}^{S} q_s z_s, \tag{6.13, 14}$$

where the supremum and the infimum are taken over all strictly positive state-price vectors q. Equivalently

$$q_u(z) = \frac{1}{\bar{r}} \sup_{\pi^*} E^*(z), \qquad q_\ell(z) = \frac{1}{\bar{r}} \inf_{\pi^*} E^*(z), \tag{6.25, 26}$$

where the supremum and the infimum are taken over all risk-neutral probabilities.

Notes: This part was based on Chapters 3, 5 and 6 from LeRoy and Werner (2001).

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Appendix: Farkas-Stiemke Lemma (6.3)

Let y and a be m-dimensional vectors, b an n-dimensional vector, and Y an $m \times n$ matrix for arbitrary m, n.

Farkas Lemma, 6.3.1: *There does not exist* $a \in \mathcal{R}^m$ *such that*

$$aY \geq 0 \text{ and } ay < 0$$

iff there exists $b \in \mathcal{R}^n$ *such that*

$$y = Yb \text{ and } \ b \geq 0.$$

With $Y = X$, $y = p$, $a = h$ and $b = q$, Farkas' Lemma says that no strong arbitrage is equivalent the existence of positive state prices.

Stiemke's Lemma, 6.3.2: *There does not exist* $a \in \mathcal{R}^m$ *such that*

$$aY \geq 0 \text{ and } ay \leq 0, \text{ with at least one strict inequality}$$

iff there exists $b \in \mathcal{R}^n$ *such that*

$$y = Yb \text{ and } \ b \gg 0.$$

With $Y = X$, $y = p$, $a = h$ and $b = q$, Stiemke's Lemma says that security prices p exclude arbitrage iff there exists a strictly positive vector of state prices.

3. Expected Utility and Multiple-Prior Expected Utility.

If there is no date-0 consumption, expected utility takes the form

$$\sum_{s=1}^{S} \pi_s v(c_s) \tag{8.2}$$

where $v : \mathcal{R} \to \mathcal{R}$ is von Neumann-Morgenstern (or Bernoulli) utility and π_s are (subjective) probabilities. Function v is unique up to an increasing linear transformation.

We write expected utility as $E[v(c)]$.

Topic omitted: axiomatizations of expected utility representation. (8.5)

If there is date-0 consumption, expected utility (separable over time) is

$$v_0(c_0) + \sum_{s=1}^{S} \pi_s v_1(c_s) \tag{8.14}$$

for $v_0 : \mathcal{R} \to \mathcal{R}$ and $v_1 : \mathcal{R} \to \mathcal{R}$. We write $v_0(c_0) + E[v_1(c_1)]$.

Marginal rates of substitution for expected utility:

Vector of MRS for expected utility (8.14) at c is

$$\left(\pi_1 \frac{v_1'(c_1)}{v_0'(c_0)}, \ \ldots, \ \pi_S \frac{v_1'(c_S)}{v_0'(c_0)} \right)$$

If c is risk-free, then MRS is proportional to the vector $(\pi_1, \ldots, \pi_S)$.

State-dependent expected utility is

$$\sum_{s=1}^{S} \pi_s v_s(c_s)$$

for S functions $v_s : \mathcal{R} \to \mathcal{R}$. Probabilities don't matter. This is **state-separable** utility function.

Ellsberg paradox:

An urn has 90 balls of which 30 are red and the rest are blue and yellow. Exact numbers of blue balls and yellow balls are not known. Consider bets of \$ 1 on a ball of a certain color (or colors) drawn from the urn. Denote bets by 1_R, 1_B, $1_{R \vee Y}$, etc. Typical preferences over bets are

$$1_R \succ 1_B, \qquad 1_{B \vee Y} \succ 1_{R \vee Y}$$

This pattern of preferences is incompatible with expected utility: it cannot be that $\pi(R) > \pi(B)$ and $\pi(B \vee Y) > \pi(R \vee Y)$, because $\pi(B \vee Y) = \pi(B) + \pi(Y)$ holds for any probability measure π.

Multiple-Prior Expected Utility.

An alternative to expected utility and one that can explain the Ellsberg paradox is the **multiple-prior expected utility.**

It takes the form

$$\min_{P \in \mathcal{P}} E_P[v(c)], \tag{8.10}$$

where $v : \mathcal{R}_+ \to \mathcal{R}$ is von Neumann-Morgenstern utility (with no date-0 consumption) and $\mathcal{P}$ is a convex and closed set of probability measures on $\mathcal{S}$.

Set of probability measures (priors) $\mathcal{P}$ reflects agent's **ambiguous beliefs**.

Examples of sets of priors:

- The set Δ of all probabilities on S. Then

$$\min_{P \in \Delta} E_P[v(c)] = \min_s v(c_s).$$

This is the maxmin utility of Hurwicz (1952).

- Bounds on probabilities:

$$\mathcal{P} = \{P \in \Delta : \lambda_s \leq P(s) \leq \gamma_s, \ \forall s\},$$

where $\lambda_s, \gamma_s \in [0, 1]$ are lower and upper bounds on probability of state s, respectively, and such that $\sum_s \lambda_s \leq 1$ and $\sum_s \gamma_s \geq 1$.

Differentiability and Marginal Rates of Substitution for MPEU.

Let $\mathcal{P}(c)$ denote the set of minimizing probabilities in (8.10) at c, that is

$$\mathcal{P}(c) = \arg\min_{P \in \mathcal{P}} E_P[v(c].$$

Suppose that v is differentiable and concave. Then multiple-prior expected utility (8.10) is differentiable if and only if $\mathcal{P}(c)$ consists of a single probability measure.

The MRS at a point of differentiability of MPEU with date-0 consumption is

$$\left(\pi_1 \frac{v_1'(c_1)}{v_0'(c_0)}, \ \ldots, \ \pi_S \frac{v_1'(c_S)}{v_0'(c_0)}\right)$$

where π is the unique minimizing probability measure in $\mathcal{P}(c)$.

If c_1 is risk-free, then MPEU is non-differentiable at c_1.

Expected utility is in Chapter 10 of LeRoy and Werner (2001)

Multiple-prior expected utility has been proposed by Gilboa and Schmeidler (1989). Also called **maxmin utility**.

4. Risk Aversion for Expected Utility

• Assume that the agent has expected utility function and there is no date-0 consumption.

The agent is **risk averse** if

$$E[v(c)] \leq v(E(c)), \tag{9.1}$$

for every consumption plan $c = (c_1, \ldots, c_S) \in \mathcal{R}^S$. We recall that $E[v(c)] = \sum_s \pi_s v(c_s)$. Also $E(c) = \sum_s \pi_s c_s$.

The agent is **strictly risk averse** if

$$E[v(c)] < v(E(c)) \tag{9.2}$$

for every nondeterministic consumption plan c, i.e., every c with $c \neq E(c)$.

The agent is **risk neutral** if

$$E[v(c)] = v(E(c)) \tag{9.3}$$

for every consumption plan c.

Concavity and Risk Aversion I (9.3)

If v is concave, then

$$E[v(c)] \leq v(E(c))$$

for every c. This is the **Jensen's inequality**. Agent with concave von Neumann-Morgenstern utility function is risk averse.

Similarly, if v is strictly concave, then

$$E[v(c)] < v(E(c))$$

for every nondeterministic c. This is the strict version of Jensen's inequality. Agent with a strictly concave von N-M utility function is strictly risk averse. Of course, if v is linear, $v(y) = y$ (or $v(y) = Ay + B$), then

$$E[v(c)] = v(E(c))$$

Agent with linear utility is risk neutral.

Measures of Risk Aversion (9.4-5)

The **risk compensation** for additional risky consumption plan z with $Ez = 0$ at deterministic "initial" consumption $w \in \mathcal{R}$ is $\rho(w, z)$ that solves

$$E[v(w + z)] = v\big(w - \rho(w, z)\big). \tag{9.11}$$

If v is twice-differentiable and strictly increasing (so that $v'(w) > 0$ for every w), we also have:

– the Arrow-Pratt measure of **absolute risk-aversion**

$$A(w) \equiv -\frac{v''(w)}{v'(w)}, \tag{9.9}$$

– the Arrow-Pratt measure of **relative risk aversion**

$$R(w) \equiv -\frac{v''(w)}{v'(w)}w. \tag{9.23}$$

The Theorem of Pratt

Let v_1, v_2 be two twice-differentiable, strictly increasing vNM. utility functions with ρ_1, ρ_2, and A_1 and A_2, respectively.

Theorem (9.6.1): *The following conditions are equivalent:*

(i) $A_1(w) \geq A_2(w)$ *for every* $w \in \mathcal{R}$.

(ii) $\rho_1(w, z) \geq \rho_2(w, z)$ *for every* $w \in \mathcal{R}$ *and every risky plan* z *with* $Ez = 0$.

(iii) v_1 *is a concave transformation of* v_2, *i.e.* $v_1 = f \circ v_2$ *for* f *concave and strictly increasing.*

Concavity and Risk Aversion (II)

Let v be twice-differentiable and strictly increasing. The Theorem of Pratt implies the following:

Corollary:

(i) *An agent is risk averse iff his von Neumann-Morgenstern utility function v is concave.*

(ii) *An agent is risk neutral iff his von Neumann-Morgenstern utility function v is linear.*

(iii) *An agent is strictly risk averse iff his von Neumann-Morgenstern utility function v is strictly concave.*

Note: "*iff*" means "if and only if."

This corollary holds true even without the assumption of differentiability of v, see Theorem 9.3.1.

Decreasing, Constant and Increasing Risk Aversion

Another implication of the Theorem of Pratt is the following:

Theorem (9.7.1):

(i) $\rho(w, z)$ *is increasing in* w *for every* z*, iff* $A(w)$ *is increasing in* w*.*

(ii) $\rho(w, z)$ *is constant in* w *for every* z*, iff* $A(w)$ *is constant in* w*.*

(iii) $\rho(w, z)$ *is decreasing in* w *for every* z *iff* $A(w)$ *is decreasing in* w*.*

Some Common Utility Functions

The functions most often used as von Neumann-Morgenstern utility functions in applied work and as examples are:

Linear utility:

$$v(y) = y$$

has zero absolute risk aversion, so the agent is risk-neutral.

Negative Exponential Utility:

$$v(y) = -e^{-\alpha y},$$

where $\alpha > 0$, has constant absolute risk-aversion equal to α.

Quadratic utility:

$$v(y) = -(\alpha - y)^2, \quad \text{for} \quad y < \alpha,$$

has absolute risk aversion equal to $1/(\alpha - y)$.

Logarithmic utility:

$$v(y) = ln(y + \alpha), \quad \text{for} \quad y > -\alpha.$$

If $\alpha = 0$, then relative risk-aversion is constant.

Power utility:

$$v(y) = \frac{y^{1-\gamma}}{1-\gamma}, \quad \text{for} \quad y \geq 0,$$

where $\gamma \geq 0, \gamma \neq 1$, has constant relative risk-aversion equal to γ.

Linear Risk Tolerance

The **risk tolerance**:

$$T(w) \equiv \frac{1}{A(w)}. \tag{9.10}$$

The negative exponential utility function, the quadratic utility function, the logarithmic utility function, the power utility function — all have linear risk tolerance (LRT or HARA).

Notes: This part was based on Chapter 10 from LeRoy and Werner (2001)

Proof of the Pratt's Theorem: (i) implies (iii): Define $f(t) = v_1(v_2^{-1}(t)$.

The derivative of f is

$$f'(t) = \frac{v_1'(v_2^{-1}(t))}{v_2'(v_2^{-1}(t))}$$

and is strictly positive since $v_i' > 0$ for $i = 1, 2$. The second derivative is

$$f''(t) = \frac{v_1''(w) - (v_2''(w)v_1'(w))/v_2'(w)}{[v_2'(w)]^2},$$

where $w = v_2^{-1}(t)$. This can be rewritten as

$$f''(t) = \big(A_2(w) - A_1(w)\big)\frac{v_1'(w)}{[v_2'(w)]^2}.$$

Thus $f'' \leq 0$, and hence f is concave.

(iii) implies (ii): By the definition of ρ,

$$E[v_1(w+z)] = v_1\big(w - \rho_1(w,z)\big).$$

Since $v_1 = f \circ v_2$ and f is concave, Jensen's inequality yields

$$E[v_1(w+z)] = E[f(v_2(w+z))] \leq f(E[v_2(w+z)]).$$

The right-hand side equals $f\big(v_2(w - \rho_2(w,z))\big)$, and so

$$v_1\big(w - \rho_1(w,z)\big) \leq v_1\big(w - \rho_2(w,z)\big).$$

Since v_1 is strictly increasing, $\rho_1(w,z) \geq \rho_2(w,z)$.

(ii) implies (i): (... in class)

5. Optimal Portfolios

- Assume that there is no date-0 consumption.

Optimal Portfolios with One Risky Security

Suppose that there are only two securities: risk-free security with return $\bar{r}$, and a single risky security with return r.

Portfolio choice problem for expected utility (8.2) can be written as

$$\max_a E[v((w-a)\bar{r}+ar)], \tag{11.10}$$

or equivalently

$$\max_a E[v(w\bar{r}+(r-\bar{r})a)],$$

where a is the dollar amount invested in risky security.

The first-order condition for an interior solution a^* to (11.10) is

$$E[v'(w\bar{r}+a^*(r-\bar{r}))(r-\bar{r})]=0. \tag{11.11}$$

Example, 11.3.1: For quadratic utility, we have

$$a^* = \frac{(\alpha - w\bar{r})(\mu-\bar{r})}{\sigma^2+(\mu-\bar{r})^2},$$

where $\mu = E(r)$ and $\sigma^2 = \text{var}(r)$.

Theorem, 11.4.1: *If an agent is strictly risk averse and has differentiable vNM utility function, then the optimal investment in the risky security is strictly positive, zero or strictly negative iff the risk premium on the risky security (i.e.,* $E(r) - \bar{r}$*) is strictly positive, zero or strictly negative.*

Optimal Portfolios with Many Risky Securities

Let security 1 be risk-free with return $\bar{r}$. Portfolio choice problem for expected utility can be rewritten

$$\max_{a_2,\ldots,a_J} E[v(w\bar{r} + \sum_{j=2}^{J} a_j(r_j - \bar{r}))]. \qquad (13.4)$$

The solution a^* is the optimal investment.

Optimal Portfolios under Fair Pricing

Theorem, 13.4.1: *The payoff of an optimal portfolio of a strictly risk-averse agent with differentiable vNM utility function is risk free iff all securities are priced fairly; that is, iff*

$$E(r_j) = \bar{r} \qquad \forall \quad j.$$

Risk-Return Tradeoff

Theorem, 13.3.1: *If* r^* *is the return on an optimal portfolio of a risk-averse agent, then* $E(r^*) \geq \bar{r}$*. Further, if the agent is strictly risk averse and* r^* *is nondeterministic, then* $E(r^*) > \bar{r}$*.*

Optimal Portfolios under Linear Risk Tolerance

Theorem, 13.6.1: *If an agent's risk tolerance is linear, $T(y) = \alpha + \gamma y$, then the optimal investment in each risky security is given by*

$$a_j^*(w) = (\alpha + \gamma w\bar{r})b_j, \qquad \text{for } j = 2, \ldots, J,$$

for some b_j which is independent of wealth and of parameter α. Hence the optimal investment in each security is a linear function of wealth.

This implies that the ratio of optimal investments in risky securities is independent of wealth for an LRT agent. Consequently, optimal investments at different levels of wealth differ only by the amounts of wealth invested in risky securities, and not by the compositions of the portfolios of risky securities.

Theorem 13.6.1 applies to negative exponential utility, logarithmic and power utilities.

Notes: This part was based on Chapters 12 and 13 from LeRoy and Werner (2001).

Optimal Portfolios for Multiple-Prior Expected Utility

Portfolio choice problem with one risky security for MPEU (8.10) is written as

$$\max_a \min_{P\in\mathcal{P}} E_P[v(w\bar{r} + a(r - \bar{r}))]. \qquad (*)$$

We have

Theorem 1: *Suppose that v is concave. If*

$$\min_{P\in\mathcal{P}} E_P[r] \le \bar{r} \le \max_{P\in\mathcal{P}} E_P[r], \qquad (**)$$

then the optimal investment in the risky security is zero.

Further

Theorem 2: *Suppose that v be strictly concave and differentiable. The optimal investment in the risky security is strictly positive iff*

$$\bar{r} < \min_{P\in\mathcal{P}} E_P[r].$$

Similarly, the optimal investment is strictly negative iff

$$\max_{P\in\mathcal{P}} E_P[r] < \bar{r}.$$

6. Equilibrium Prices in Security Markets

An **equilibrium** in security markets consists of a vector of security prices p, a portfolio allocation $\{h^i\}$, and a consumption allocation $\{(c_0^i, c_1^i)\}$ such that

(1) Consumption plan (c_0^i, c_1^i) and portfolio h^i are a solution to agent i's consumption-portfolio choice problem (1.4) at prices p,

(2) Markets clear:

$$\sum_i h^i = 0,$$

and

$$\sum_i c_0^i \leq \bar{w}_0, \qquad \sum_i c_1^i \leq \bar{w}_1.$$

Remarks: (1) agents take prices as given; (2) they agree on what states may occur, and on what payoffs are associated with each state.

Existence of Equilibrium. (1.8)

Theorem, 1.8.1: *If each agent's admissible consumption plans are restricted to be positive, his utility function is strictly increasing and quasi-concave, his initial endowment is strictly positive, and there exists a portfolio with positive and nonzero payoff, then there exist an equilibrium is security markets.*

Consumption-Based Security Pricing

For an agent with expected utility function

$$v_0(c_0) + E[v_1(c_1)] \qquad (*)$$

first-order conditions (1.5) for optimal consumption (c_0, c_1) (interior) are

$$p_j = \sum_{s=1}^{S} \pi_s \frac{v_1'(c_s)}{v_0'(c_0)} x_{js} \qquad \forall\ j \qquad (14.1)$$

or, using expectation,

$$p_j = \frac{E[v_1'(c_1)x_j]}{v_0'(c_0)} \qquad \forall\ j. \qquad (14.1')$$

For risk-free security with return $\bar{r}$, this FOC implies

$$\bar{r} = \frac{v_0'(c_0)}{E[v_1'(c_1)]}. \qquad (14.3)$$

For risky security j, we obtain from (14.1') and (14.3)

$$E(r_j) = \bar{r} - \bar{r}\frac{cov(v_1'(c_1), r_j)}{v_0'(c_0)}. \qquad (14.6)$$

(14.6) is the equation of **Consumption-Based Security Pricing** (CSBP).

CBSP holds for any portfolio return r:

$$E(r) = \bar{r} - \bar{r}\frac{cov(v_1'(c_1), r)}{v_0'(c_0)}. \qquad (14.7)$$

Two contingent claims y and z are **co-monotone** if $y_s \geq y_t$ iff $z_s \geq z_t$ for all states s and t. Analogously, y and z are **negatively co-monotone** iff $y_s \geq y_t$ iff $z_s \leq z_t$, for all states s and t. Note that if two pairs y and z, and z and w are co-monotone, then y and w must be co-monotone.

Proposition: *If y and z are co-monotone, then cov$(z, y) \geq 0$. If in addition y and z are nondeterministic, then cov$(z, y) > 0$.*

Proof: This follows from

$$\operatorname{cov}(y, z) = \frac{1}{2} \sum_{s=1}^{S} \sum_{t=1}^{S} \pi_s \pi_t (y_s - y_t)(z_s - z_t).$$

Combining this Proposition with CBSP, we obtain

Theorem: *If an agent is risk averse, then $E(r) \geq \bar{r}$ for every return r that is co-monotone with optimal consumption. For every return r that is negatively co-monotone with optimal consumption, it holds $E(r) \leq \bar{r}$.*

Volatility of Marginal Rates of Substitution

The first-order condition for expected utility (∗) also implies that

$$\sigma\left(\frac{v_1'(c_1)}{v_0'(c_0)}\right) \geq \frac{|E(r_j) - \bar{r}|}{\bar{r}\sigma(r_j)}. \tag{14.15}$$

where $\sigma(\cdot)$ denotes the standard deviation.

The ratio of risk premium to standard deviation of return is called the **Sharpe ratio**. The marginal rate of substitution between consumption at date 0 and at date 1 in equilibrium is higher than the (absolute value of) the Sharpe ratio of each security divided by the risk-free return.

Inequality (14.15) is known as Hansen-Jagannathan bound.

Further,

$$\sigma\left(\frac{v_1'(c_1)}{v_0'(c_0)}\right) \geq \sup_r \frac{|E(r) - \bar{r}|}{\bar{r}\sigma(r)}. \tag{14.16}$$

Notes: This part was based on Chapter 14 of LeRoy and Werner (2001).

Equilibrium: An Example.

Remark: If there is no date-0 consumption, then first-order conditions (1.5) for an interior optimal consumption under expected utility are

$$\lambda_0^i p_j = \sum_{s=1}^{S} \pi_s v'^i(c_{1s}^i) x_{js}, \qquad j = 1, \ldots, K,$$

where λ_0^i is agent i's Lagrange multiplier associated with date-0 budget constraint.

Example: Two states, $s = 1, 2$; two agents, $i = 1, 2$. No consumption at date 0. Agents have the same expected utility function with $v^i(y) = \sqrt{y}$ and probability 1/2 for each state. Expectd utility is

$$\frac{1}{2}\sqrt{c_1} + \frac{1}{2}\sqrt{c_2}.$$

They are strictly risk averse.

Date-1 endowments are $\omega^1 = (1, 3)$ and $\omega^2 = (3, 1)$; there are no endowments at date 0. Note that the aggregate endowment is state independent. Agents are risk averse.

There are two securities with payoffs $x_1 = (1, 1)$ and $x_2 = (2, 1)$.

We show that the equilibrium consumption allocation is

$$c^1 = (2, 2), \qquad c^2 = (2, 2),$$

i.e., risk-free allocation. We need to find equilibrium asset prices and portfolios, and verify that they indeed are an equilibrium in security markets.

First, we find security prices using the first-order conditions. The right hand side of the FOC for agent 1 is $\frac{1}{2\sqrt{2}}$ for security 1 and $\frac{3}{4\sqrt{2}}$ for security 2. The same values obtain for agent 2. If we set prices $p_1 = 1$ and $p_2 = 3/2$, we have the first-order conditions satisfied for both agents with multipliers $\lambda_0^1 = \lambda_0^2 = \frac{1}{2\sqrt{2}}$.

Next we find portfolios. They must be such that the payoff of agent i's portfolio $h^i = (h_1^i, h_2^i)$ equals his net trade $c^i - \omega^i$. We obtain

$$h^1 = (-3, 2), \qquad h^2 = (3, -2).$$

These portfolios clear the security markets: $h^1 + h^2 = 0$. Further, they satisfy date-0 budget constraint $ph^i \leq 0$ at prices $p = (1, 3/2)$.

Summing up, market clearing conditions, budget constraints, and the first-order conditions are all satisfied. Utility functions are concave, so first-order conditions are sufficient. We conclude that p, h^1, h^2, c^1, c^2 are an equilibrium.

At equilibrium prices $p_1 = 1$ and $p_2 = 3/2$, security returns are $\bar{r} = 1$ and $r_2 = (4/3, 2/3)$. It holds $E(r_2) = \bar{r}$ in accordance with CBSP.

7. Pareto-Optimal Allocations of Risk

Consumption allocation $\{\tilde{c}^i\}$ *Pareto dominates* another allocation $\{c^i\}$ if every agent i weakly prefers consumption plan $\tilde{c}^i$ to c^i, that is,

$$u^i(\tilde{c}^i) \geq u^i(c^i),$$

and in addition at least one agent i strictly prefers $\tilde{c}^i$ to c^i (so that strict inequality holds for at least one i).

A feasible consumption allocation $\{c^i\}$ is **Pareto optimal** if there does not exist an alternative feasible allocation $\{\tilde{c}^i\}$ that Pareto dominates $\{c^i\}$. Feasibility of $\{c^i\}$ means that

$$\sum_{i=1}^{I} c^i \leq \bar{w},$$

where $\bar{w} = \sum_{i=1}^{I} w^i$ denotes the aggregate endowment.

If $\{c^i\}$ is interior and utility functions are differentiable, the first-order conditions for Pareto optimality are

$$\frac{\partial_s u^i(c^i)}{\partial_t u^i(c^i)} = \frac{\partial_s u^k(c^k)}{\partial_t u^k(c^k)} \qquad \forall i, k, \ \forall s, t \tag{15.6}$$

First Welfare Theorem in Complete Security Markets

Theorem, 15.3.1: *If security markets are complete and agents' utility functions are strictly increasing, then every equilibrium consumption allocation is Pareto optimal.*

Complete Markets and Options

If there is payoff $z \in \mathcal{M}$ which takes different values in different states, then $S-1$ options on z complete the markets.

Pareto-Optimal Allocations under Expected Utility

Suppose that agents' utility functions have expected utility representations with common probabilities.

Theorem, 15.5.1: *If agents are strictly risk averse, then at every Pareto-optimal allocation their date-1 consumption plans are co-monotone with the aggregate endowment.*

Of course, consumption plans are co-monotone with aggregate endoment iff they are co-monotone with each other.

The proof of Proposition 15.5.1 draws on the concept of greater risk (Ch. 10). An easier argument is available when agents' utility functions are differentiable. It applies to interior allocations. The argument is as follows:

Pareto optimal allocation $\{c^i\}$ must be a solution to the maximization of weighted sum of utilities $\sum_{i=1}^{I} \mu^i u^i(\tilde{c}^i)$ subject to feasibility $\sum_{i=1}^{I} \tilde{c}^i \leq \bar{w}$, for some weights $\mu^i > 0$. If the allocation is interior and u^i has expected utility form (∗) with the same probabilites, then first-order conditions for this constrained maximization imply that

$$\mu^i {v'}_1^i(c_s^i) = \mu^k {v'}_1^k(c_s^k), \qquad \forall i, k, \ \forall s.$$

Since date-1 marginal utilities ${v'}_1^i$ are strictly decreasing, it follows that if

$$c_s^i \geq c_t^i$$

for some i, for states s and t, then

$$c_s^k \geq c_t^k$$

for every k. Hence c_1^i and c_1^k are comonotone for every i and k.

Corollary, 15.5.2: *If agents are strictly risk averse and the aggregate date-1 endowment is state independent for a subset of states, then each agent's date-1 consumption at every Pareto-optimal allocation is state independent for that subset of states.*

Pareto-Optimal Allocations under Linear Risk Tolerance

Theorem, 15.6.1: *If every agent's risk tolerance is linear with common slope* γ*, i.e.,*

$$T^i(y) = \alpha^i + \gamma y,$$

then date-1 consumption plans at any Pareto-optimal allocation lie in the span of the risk-free payoff and the aggregate endowment.

The consumption set of agent i is $\{c \in \mathcal{R}^S : T^i(c_s) > 0, \text{ for every } s\}$.

Notes: This part was based on Chapter 15 of LeRoy and Werner (2001).

Proof of Theorem 15.3.1: Let p and $\{c^i\}$ be an equilibrium in complete security markets. Consumption plan c^i maximizes $u^i(c_0, c_1)$ subject to

$$c_0 \leq w_0^i - qz$$

$$c_1 \leq w_1^i + z, \quad z \in \mathcal{M} = \mathcal{R}^S,$$

where q is the (unique) vector of state prices.

The above budget constraints are equivalent to a single budget constraint

$$c_0 + qc_1 \leq w_0^i + qw_1^i.$$

Suppose that allocation $\{c^i\}$ is not Pareto optimal, and let $\{\tilde{c}^i\}$ be a feasible Pareto dominating allocation. Then (since u^i is strictly increasing)

$$\tilde{c}_0^i + q\tilde{c}_1^i \geq w_0^i + qw_1^i$$

for every i, with strict inequality for agents who are strictly better-off. Summing over all agents, we obtain

$$\sum_{i=1}^{I} \tilde{c}_0^i + \sum_{i=1}^{I} q\tilde{c}_1^i > \bar{w}_0 + q\bar{w}_1$$

which contradicts the assumption that allocation $\{\tilde{c}^i\}$ is feasible.