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WORKING PAPERS

## Lévy processes on the cryptocurrency market

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**Abstract:** Lévy processes are very often used in financial modelling since they address various characteristics of financial data. One of those characteristics is the heavy-tailedness of probability density functions - a very common empirical stylized fact on the cryptocurrency market. The aim of this study was to determine which type of Lévy motion fits the data of cryptocurrencies better, namely Alpha-Stable distribution or one of distributions from the family of generalized hyperbolic motions. The log-returns of 227 cryptocurrencies, standardized by the realized volatility estimated with the GARCH (1,1), were fitted to 11 types of distributions. The results show that the generalized hyperbolic motions fit the cryptocurrency data much more accurately than the Alpha-Stable distribution, similarly as in the case of TOP100 NASDAQ stocks. In the further stage of the analysis, it is shown how the distribution of cryptocurrency data varies over time, i.e. before, during, and after the 'boom-period' of 2017/2018.

**Keywords:** cryptocurrency market, distribution fitting, Generalized Hyperbolic distribution, Alpha-Stable distribution, Lévy process

**JEL codes:** C10, C30, G15

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## 1. Introduction

The literature concerning the cryptocurrency market is ongoingly expanding (Corbet et al. 2018). However, the fundamental empirical facts about the behavior of cryptocurrency time series have not been yet fully explored. The most prominent studies indicate that cryptocurrencies exhibit such empirical stylized facts as long memory, leverage effect, stochastic volatility and heavy tails (Osterrieder et al. 2016; Phillip et al. 2018; Catania et al. 2018). The tail behavior of cryptocurrency time series has been further assessed with the application of the Extreme Value Theory, emphasizing a very high risk of investment in the cryptocurrency market (Osterrieder & Lorenz 2017; Gkillas & Katsiampa 2018; Borri, 2019).

The fact that cryptocurrency time series are heavy tailed is also a reason for conducting the research presented in this paper. The study is inspired by the analysis conducted by Chan et al. (2017), Kakinaka & Umeno (2018) as well as Málek & Tran (2018). Chan et al. (2017) fit the data of seven cryptocurrencies to seven types of probability density functions (pdfs) and indicate that five of the considered cryptocurrencies follow the Generalized Hyperbolic motion, i.e. two follow the empirical Generalized Hyperbolic (GH) distribution and three follow the empirical Normal Inverse Gaussian (NIG) distribution. On the other hand, Kakinaka & Umeno (2018) fit the high-frequency data of 5 cryptocurrencies to the Lévy stable distribution (also referred to as the Alpha-Stable distribution). Their results indicate that cryptocurrencies follow the empirical Alpha-Stable distribution. Málek & Tran (2018) fit the data of four cryptocurrencies to the Alpha-Stable distribution and the Normal Inverse Gaussian distribution. Their results indicate that the empirical Alpha-Stable distribution fits the cryptocurrency data better than the NIG one.

The research presented in this paper attempts to resolve the disparity of these results. More specifically, the daily data on 227 cryptocurrencies is fitted to 11 types of probability density functions, while distinguishing between two major types of Lévy processes: the Alpha-Stable and the Generalized Hyperbolic (and its 4 other subclasses, both symmetric and asymmetric ones). Namely, this study attempts to determine whether a more heavy-tailed distribution fits the cryptocurrency data better, i.e. the Alpha-Stable distribution, or rather one of the distribution from the generalized hyperbolic family of distributions, indicating a semi-heavy-tailedness of data.

The financial data in general is usually characterized by heavy tails. Therefore, Mandelbrot (1963) started an argument that a stable Lévy process (specifically in its simplest form – the Alpha-Stable distribution) fits the time series of financial asset returns much more accurately than the Brownian motion (Bachelier 1900). Since then, scientists have developed new classes of Lévy motion, particularly the Generalized Hyperbolic distribution (Barndoff-Nielsen 1977) and its subclasses which have been accommodated to finance, such as the Hyperbolic (Eberlein & Keller 1995), the Normal Inverse Gaussian (Barndoff-Nielsen 1995, 1997) and the Variance Gamma (Madan & Seneta 1990) distributions. The generalized hyperbolic family of distributions is similar to the Alpha-Stable distribution since both are defined by four parameters: localization, asymmetry, dispersion and kurtosis. One of the main differences between those two classes of Lévy motion is that Generalized Hyperbolic distributions have less heavy tails (often referred to as 'semi-heavy') compared to the Alpha-Stable distribution (Hammerstein 2010; Walter 2016).

## 2. Data and empirical procedure

The data on daily cryptocurrency prices (denominated in USD) was obtained from coinmarketcap.com with the use of the 'crypto' package in the R software. Therefore, from the top-ranked 1500 cryptocurrencies (in terms of market capitalization, as of 02.02.2019 2:30 p.m. (GMT+1)) those which were listed (at least) since 01.09.2016 until 01.02.2019 were selected. Therefore, 238 cryptocurrencies were selected from which 11 (*Espers, FedoraCoin, Gapcoin, IncaKoin, LanaCoin, PWR_Coin, SounDAC, Sprouts, StrongHands, Tether, Uniform_Fiscal_Object*) were dropped because during the empirical distribution fitting procedure the algorithm produced errors of different types. All missing values were filled with the last previous non-missing value.

The analysis was carried out in the following manner:

1. Calculation of daily log-returns ($Z_{i,t}$), the estimation of the GARCH (1,1) model in order to calculate the realized volatility ($\sqrt{h_{i,t}}$) , and the standardization of log-returns by the realized volatility ($L_{i,t}$) (Andersen et al. 2000)

$$Z_{i,t} = \ln(X_{i,t}) - \ln(X_{i,t-1})$$

$$L_{i,t} = \frac{Z_{i,t}}{\sqrt{h_{i,t}}}$$

where $X_{i,t}$ is a daily price on day $t$ of $i$-th cryptocurrency

2. Fitting the considered empirical distributions (the Alpha-Stable, the Generalized Hyperbolic (GH), the Hyperbolic (H), the Normal Inverse Gaussian (NIG), the Variance Gamma (VG), and the skewed Student-t (T)[1], where all of the distributions except for the Alpha-Stable have two variants: symmetric and asymmetric[2]) to the data ($L_{i,t}$), using the maximum likelihood method, and extracting the log-likelihood of optimal parametrization for each distribution[3]
3. Calculation of information criteria (AIC, BIC, CAIC, AICc, HQC)[4] for the maximum log-likelihood obtained from each estimation of the optimal parameters

$$AIC = 2 * k - 2 * \ln(L)$$

$$BIC = k * \ln(n) - 2 * \ln(L)$$

$$CAIC = -2 * \ln(L) + k * (\ln(n) + 1)$$

$$AICc = AIC + \frac{2 * k * (k + 1)}{n - k - 1}$$

$$HQC = -2 * \ln(L) + 2 * k * \ln(\ln(n))$$

where: L is the maximum Likelihood obtained from parametrization, k is the number of parameters estimated, and n is the number of observations

4. The selection of best-fitting empirical distributions with respect to each criterium
5. The analysis of results where the best fitting empirical distributions are unequivocally indicated by Information criteria (all five criteria indicate the same distributions as best-fitting)

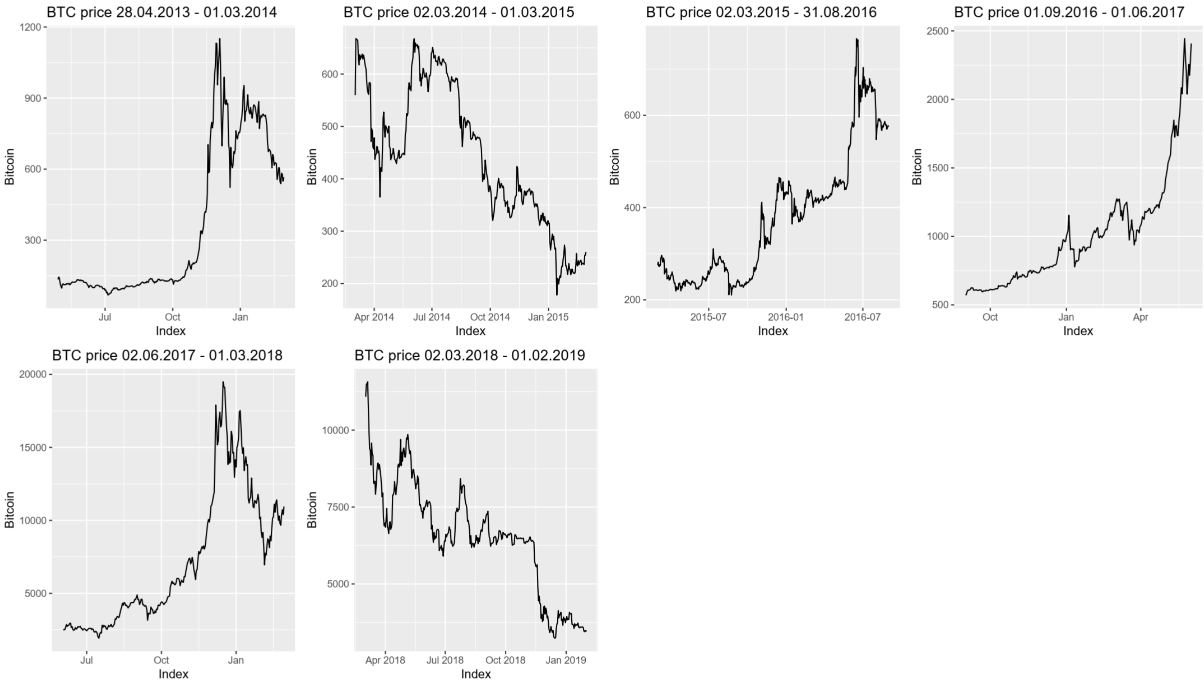
The described procedure was applied to four data samples, i.e. the entire period 01.09.2016 – 01.02.2019 and three sub-periods: the 'pre-boom' period of 01.09.2016 – 01.06.2017, the 'during-boom' period of 02.06.2017 – 01.03.2018, and the 'after-boom' period

[1] The technical properties of the considered probability density functions are well-described by Chan et al. (2017), Kakinaka & Umeno (2018) & Málek and Tran (2018). Therefore, such a description is not included in this paper.
[2] In each case the asymmetric distribution has a skewness-parameter 'gamma' estimated, while in the case of the symmetric distribution this parameter is not estimated
[3] The Generalized Hyperbolic distributions are fitted with the use of the 'ghyp' package in the R software, where the optimization routine 'optim' is used to maximize the loglikelihood function; the Alpha-Stable distribution is fitted with the use of the 'MASS' package, where the optimization routine 'optim' is used to maximize the loglikelihood function as well. The probability density function for the Alpha-Stable distribution is obtained from the 'stabledist' package, consistent with the parametrization of Nolan (2001), and modified in a way that there is no constraint on the parameters
[4] As in Chan et al. (2017)

of 02.03.2018 – 01.02.2019. The 'boom' expression refers to the period of an increased attention around the cryptocurrency market, and blockchain technology in general, which resulted in the increase of the cryptocurrency market capitalization from USD 18 billion at the beginning of 2017, to USD 100 billion at the beginning of June 2017, up to 800 billion during the peak at the beginning of 2018, followed by a decrease to USD 300 billion at the beginning of March 2018 and a further decrease to ca. 100 billion at the beginning of 2019 (coinmarketcap.com).

The selection of such sub-periods is based on the preliminary analysis of Bitcoin price trends (Fig. 1). It can be noticed that the Bitcoin price trends in periods 28.04.2013 – 01.03.2014 (the leftmost graph in the first row of Fig. 1) and 02.03.2014 – 01.03.2015 (the second graph from the left in the first row of Fig. 1) are very similar to the trends in periods 02.06.2017 – 01.03.2018 (the leftmost graph in the second row of Fig. 2) and 02.03.2018 – 01.02.2019 (the second graph from the left in the second row of Fig. 1), respectively. Therefore, it would be interesting to see in the near future, whether the Bitcoin price trend from the period 02.03.2015 – 01.06.2017 (the two rightmost graphs in the first row of Fig. 1) will repeat as well.

Moreover, since Chan et al. (2017), Kakinaka & Umeno (2018) and Málek & Tran (2018) have all proved the relevance of fitting Lévy distributions to the cryptocurrency data, it is assumed that cryptocurrency returns follow one of the Lévy motions. Therefore, tests such as the chi-squared test for goodness of fit, enabling to determine whether a particular time series follows a particular empirical distribution, is not conducted in this study. Nevertheless, the results of such tests as well as graphs comparing the observed distribution to the expected distribution, the so-called qqplots, or any other more detailed results (including the programming code used for computations) are available on the reader's request.




source: own preparation based on coinmarketcap.com

**Figure 1. The evolution of Bitcoin price, divided into 6 consequent periods (28.04.2013 – 01.02.2019)**

## 3. Empirical results of the distribution fitting

The cryptocurrency log-returns are clearly leptokurtic (Table 1), indicating their heavy-tailedness. Therefore, according to Andersen et al. (2000) it is reasonable to standardize such returns by the realized volatility estimated with the GARCH (1,1) model. Moreover, such standardization enables to remove as much predictability (i.e. conditional variance) as possible, making further results more adequate. It can be noticed that the standardized log-returns are characterized by the lower volatility, indicating that the conditional volatility has a considerable share in the overall volatility of cryptocurrency returns. Furthermore, it is surprising that during the potentially more volatile second sub-period (02.06.2017 – 01.03.2018) the volatility is in fact lower than in the case of the first sub-period (01.09.2016 – 01.06.2017). In the case of the standardized returns, the average skewness and the average kurtosis is lower in the second sub-period than in the first and third sub-period. The results of the empirical distribution fitting indicate that the most common empirical distribution on the cryptocurrency market is a symmetric Hyperbolic distribution (Table 3). This is the case not only for the entire analyzed period but also for all three sub-periods. In order to make the obtained results more meaningful, the same procedure has been applied to the TOP100 NASDAQ stocks. The results show that also in the case of the NASDAQ stock market, the most common empirical distribution is a symmetric Hyperbolic distribution.

Moreover, it can also be noticed that in the second sub-period, i.e. the time of the 'cryptocurrency-boom', the second most common empirical distribution is an Alpha-Stable distribution, indicating much bigger heavy-tailedness of cryptocurrency returns than in other periods. Such results are intuitive since in such an unstable period, full of extreme daily increases and decreases of cryptocurrencies' prices, large heavy-tailedness is well-justified.

**Table 1. Descriptive statistics for log-returns of cryptocurrency prices**

| | Minimum | 1st Quartile | Median | Mean | 3rd Quartile | Maximum |
|---|---|---|---|---|---|---|
| *Entire period 01.09.2016 – 01.02.2019* | | | | | | |
| **Mean** | -0.014 | -0.0001 | 0.002 | 0.001 | 0.003 | 0.014 |
| **Std. dev.** | 0.043 | 0.1 | 0.133 | 0.161 | 0.196 | 0.706 |
| **Skewness** | -12.090 | 0.284 | 0.670 | 0.78 | 1.203 | 12.875 |
| **Kurtosis** | 1.64 | 6.555 | 10.823 | 23.48 | 18.87 | 321.89 |
| *First sub-period 01.09.2016 – 01.06.2017* | | | | | | |
| **Mean** | -0.032 | 0.004 | 0.007 | 0.007 | 0.011 | 0.023 |
| **Std. dev.** | 0.032 | 0.101 | 0.139 | 0.183 | 0.24 | 1.046 |
| **Skewness** | -6.906 | 0.21 | 0.706 | 0.864 | 1.362 | 9.447 |
| **Kurtosis** | 0.025 | 3.999 | 7.265 | 14.762 | 15.828 | 128.44 |
| *Second sub-period 02.06.2017 – 01.03.2018* | | | | | | |
| **Mean** | -0.007 | 0.003 | 0.005 | 0.005 | 0.008 | 0.023 |
| **Std. dev.** | 0.021 | 0.112 | 0.136 | 0.17 | 0.192 | 0.697 |
| **Skewness** | -8.683 | 0.153 | 0.609 | 0.688 | 1.147 | 4.954 |
| **Kurtosis** | 0.472 | 2.803 | 4.995 | 7.894 | 9.531 | 115.61 |
| *Third sub-period 02.03.2018 – 01.02.2019* | | | | | | |
| **Mean** | -0.025 | -0.008 | -0.007 | -0.007 | -0.005 | 0.01 |
| **Std. dev.** | 0.017 | 0.068 | 0.092 | 0.115 | 0.138 | 0.482 |
| **Skewness** | -6.931 | -0.142 | 0.154 | 0.394 | 0.650 | 16.914 |
| **Kurtosis** | 0.298 | 2.304 | 4.087 | 10.161 | 8.913 | 299.31 |

**Table 2. Descriptive statistics for standardized log-returns of cryptocurrency prices**

| | Minimum | 1st Quartile | Median | Mean | 3rd Quartile | Maximum |
|---|---|---|---|---|---|---|
| *Entire period 01.09.2016 – 01.02.2019* | | | | | | |
| **Mean** | 0.040 | 0.095 | 0.124 | 0.150 | 0.183 | 0.596 |
| **Std. dev.** | 0.004 | 0.037 | 0.06 | 0.081 | 0.096 | 0.471 |
| **Skewness** | -1.412 | 1.641 | 2.677 | 3.386 | 4.046 | 24.322 |
| **Kurtosis** | -1.076 | 2.966 | 9.806 | 27.559 | 22.233 | 653.973 |

| | | | | | | |
|---|---|---|---|---|---|---|
| *First sub-period 01.09.2016 – 01.06.2017* | | | | | | |
| **Mean** | 0.0307 | 0.093 | 0.133 | 0.167 | 0.21 | 0.909 |
| **Std. dev.** | 0.0034 | 0.033 | 0.057 | 0.087 | 0.106 | 0.693 |
| **Skewness** | -0.5656 | 1.333 | 2.173 | 2.598 | 3.104 | 13.867 |
| **Kurtosis** | -1.214 | 1.385 | 5.812 | 13.07 | 12.35 | 208.176 |
| *Second sub-period 02.06.2017 – 01.03.2018* | | | | | | |
| **Mean** | 0.0242 | 0.107 | 0.135 | 0.166 | 0.196 | 0.739 |
| **Std. dev.** | 0.0008 | 0.030 | 0.05 | 0.069 | 0.083 | 0.375 |
| **Skewness** | 0.4741 | 1.095 | 1.829 | 2.090 | 2.798 | 8.944 |
| **Kurtosis** | -1.207 | 1.068 | 3.887 | 8.095 | 10.439 | 100.06 |
| *Third sub-period 02.03.2018 – 01.02.2019* | | | | | | |
| **Mean** | 0.037 | 0.078 | 0.102 | 0.124 | 0.151 | 0.491 |
| **Std. dev.** | 0.0002 | 0.018 | 0.029 | 0.045 | 0.058 | 0.294 |
| **Skewness** | -0.58 | 1.18 | 2.006 | 2.468 | 3.062 | 18.177 |
| **Kurtosis** | -0.973 | 1.413 | 5.833 | 13.163 | 12.330 | 329.59 |

The last part of the empirical study contains the analysis of the consistency of the results of the distribution fitting for all of the considered periods (Table 4). The results indicate that the fitted empirical distribution is the same in all of the analyzed periods in the case of 28 cryptocurrencies (Table 4 – the first row). Moreover, there are no cryptocurrencies for which the empirical distribution is the same in the entire period, the second and the third sub-period but different in the first sub-period (Table 4 – the fourth row). It can also be noticed that the consistency between the entire period and one of the sub-periods is the highest in the case of the third sub-period (Table 4 – the seventh row) and the lowest in the case of the second sub-period (Table 4 – the sixth row). Last but not least, a very interesting finding can be noticed in the eleventh row of Table 4. More specifically, if the fitted empirical distribution is the same in the first sub-period and the third sub-period but different in the second sub-period the cryptocurrency follows the Alpha-Stable distribution in all cases. This means that in all of those twenty cases, the cryptocurrency followed a particular empirical distribution in the first sub-period (in most cases the symmetric Hyperbolic distribution) then followed the Alpha-Stable distribution in the second sub-period and then in the third sub-period it followed again the same empirical distribution as in the first sub-period.

**Table 3. Summary of the distribution fitting - only considering the series for which the selection of the distribution is consistent within all 5 information criteria**

| Alpha-Stable distribution | Symmetric generalized hyperbolic motions | | | | | Asymmetric generalized hyperbolic motions | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | GHYP | HYP | NIG | VG | T | GHYP | HYP | NIG | VG | T |
| *Cryptocurrencies, period 01.09.2016 – 01.02.2019)* *N = 214* | | | | | | | | | | |
| **9** | **6** | **131** | **1** | **53** | **5** | **1** | **6** | **2** | **0** | **0** |
| *Cryptocurrencies, period 01.09.2016 – 01.06.2017 (sub-period 1)* *N = 192* | | | | | | | | | | |
| **11** | **11** | **111** | **2** | **19** | **8** | **9** | **9** | **1** | **1** | **10** |
| *Cryptocurrencies, period 02.06.2017 – 01.03.2018 (sub-period 2)* *N = 191* | | | | | | | | | | |
| **47** | **8** | **79** | **2** | **14** | **8** | **4** | **11** | **1** | **2** | **15** |
| *Cryptocurrencies, period 02.03.2018 – 01.02.2019 (sub-period 3)* *N = 189* | | | | | | | | | | |
| **12** | **18** | **109** | **1** | **26** | **4** | **3** | **8** | **3** | **2** | **3** |
| *NASDAQ TOP100 stocks, period 01.09.2016 – 01.02.2019* *N = 97* | | | | | | | | | | |
| **8** | **6** | **49** | **0** | **13** | **6** | **3** | **4** | **1** | **1** | **6** |

Notation: N denotes the number of cryptocurrencies in the particular period for which all five information criteria indicated the same empirical distribution as the best-fitting one.

**Table 4. The consistency of the empirical distribution fitting among the analyzed periods**

| Periods | Number of cryptocurrencies | Comments |
|---|---|---|
| TOTAL = P1 = P2 = P3 | **28** | |
| TOTAL = P1 = P2 | **50** | |
| TOTAL = P1 = P3 | **41** | |
| TOTAL = P2 = P3 | **28** | The same cryptocurrencies as for TOTAL=P1=P2=P3 |
| TOTAL = P1 | **72** | Where (TOTAL = P1) ≠ P2 ≠ P3: **9** |
| TOTAL = P2 | **50** | The same cryptocurrencies as for TOTAL = P1 = P2 |
| TOTAL = P3 | **85** | Where (TOTAL = P3) ≠ P1 ≠ P2: **44** |
| TOTAL = (P1 or P2 or P3) | **116** | |
| P1 = P2 = P3 | **39** | |
| P1 = P2 | **138** | Where (P1 = P2) ≠ TOTAL: **88** |
| P1 = P3 | **59** | Where (P1 = P3) ≠ P2: **20** (all these 20 cryptocurrencies follow empirical Alpha-Stable distribution in the second period) |
| P2 = P3 | **42** | |

Notation: TOTAL denotes the entire analyzed period of 01.09.2016 – 01.02.2019, while P1, P2, and P3 denote the first, second, and third sub-periods, respectively. For instance, TOTAL = P1 = P2 denotes how many cryptocurrencies followed the same empirical distribution in the entire period as well as in the first and second sub-peiords. On the other hand, (TOTAL = P1) ≠ P2 ≠ P3 denotes how many cryptocurrencies followed the same empirical distribution in the entire period and in the first sub-period, while another in the second and third sub-periods.

## 4. Conclusion

The results of the empirical distribution fitting exercise presented in this paper indicate that cryptocurrencies tend to follow empirical distributions characterized by semi-heavy tails, i.e. one of the distributions from the Generalized Hyperbolic family (especially a symmetric Hyperbolic distribution), rather than a strongly heavy-tailed distribution, i.e. the Alpha-Stable distribution. This results are in line with the results obtained for the TOP100 NASDAQ stocks.

However, in the second analyzed sub-period of 02.06.2017 – 01.03.2018, which was the period of multiple large turbulences on the cryptocurrency market, the Alpha-Stable distribution became a much more common empirical distribution than in other periods. Such a result is reasonable since large turbulences in the financial market cause the heavy tails in the empirical distribution. In fact, there were twenty cases of cryptocurrencies which followed an Alpha-Stable distribution in the second sub-period and then in the third sub-period followed again a distribution which they followed in the first sub-period.

The results obtained in this study provide arguments to the discussion about the empirical distribution of cryptocurrency time series. The main conclusion is that cryptocurrencies behave much rather like semi-heavy tailed Generalized Hyperbolic Lévy processes than the heavy-tailed Lévy stable processes. However, in the more turbulent periods, such as the 02.06.2017 – 01.03.2018 period, the Lévy stable process is more common on the cryptocurrency market.

It is worth noticing that most of the cryptocurrencies considered in this study are 'pure cryptocurrencies', which means that they operate on their own blockchain (since there were not many other types of cryptocurrencies available on the market before 01.09.2016). Therefore, the applied procedure may be used also in the case of tokens issued through the ICO process, which often operate on the Ethereum blockchain.

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Data sources:

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## Appendix

### Table 5. The results of the distribution fitting for time series where all 5 criteria indicated the same distribution as best-fitting

| Alpha-Stable distribution | Symmetric generalized hyperbolic motions | | | | | Asymmetric generalized hyperbolic motions | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | GHYP | HYP | NIG | VG | T | GHYP | HYP | NIG | VG | T |
| *Cryptocurrencies, period 01.09.2016 – 01.02.2019)* <br> *N = 214* | | | | | | | | | | |

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|---|---|---|---|---|---|---|---|---|---|---|
| Advanced Internet Block; GCN Coin ; Memetic PepeCoin; Nullex; PetroDollar; Shift: Viacoin; WhiteCoin; Yocoin | Dimecoin; HunterCoin ; Megacoin; Mintcoin; PopularCoin; TittieCoin | Adzcoin; Anoncoin; Argentum; AudioCoin; Augur; Auroracoin; Bata; Bela; BitBar; bitBTC; Bitcoin Fast; Bitcoin Plus; BitCrystals; ; bitCNY; bitEUR; bitGold; Bitmark; BitSend; bitSilver; Bitstar; bitUSD; Bitzeny; Blocknet; Bolivarcoin; Breakout; Burst; Bytecoin; Canada_eCoin; CannabisCoin; Capricoin; CasinoCoin; Circuits_of_Value; Clams; CloakCoin; Cryptonite; Curecoin; Deutsche_eMark; DigiByte; Digitalcoin; DigitalNote; DNotes; Dogecoin; Dotcoin; E_Dinar_Coin; e_Gulden; Elcoin; Elementrem; Elite; EverGreenCoin; ExclusiveCoin; FairCoin; Fastcoin; FLO; FoldingCoin; Freicoin; FujiCoin; GeoCoin; Global Currency Reserve; GlobalBoost Y; GoldCoin; GridCoin; Gulden; HEAT; HempCoin; HiCoin; HOdlcoin; HyperStake; I_O_Coin; ION; Joulecoin; Kobocoin; Kore; LEOcoin; Magi; Manna; MarteXcoin; Moin; MonaCoin; MonetaryUnit; Myriad; Namecoin; NavCoin; NEM; Nexus; NobleCoin; Novacoin; NuBits; OBITS; Omni; Orbitcoin; PACcoin; PayCoin; Pesetacoin; PinkCoin; PIVX; PotCoin; PutinCoin; Ratecoin; ReddCoin; RevolutionVR; Rimbit; Rise; Safe Exchange Coin; SaluS; SIBCoin; SixEleven; SmileyCoin; SolarCoin; Sphere; SpreadCoin; Startcoin; Stealth; Steem_Dollars; Stellar; Stratis; Syscoin; | I0Coin | ArtByte; AurumCoin; Bean_Cash; BitBay; Bitswift; BlackCoin; BlueCoin; Breakout_Stake; Bullion; ChessCoin; Creditbit; Crown; Dash; Decred; Diamond; DigitalPrice; DigixDAO; DopeCoin; EDRCoin; Einsteinium; Emercoin; Ethereum; Ethereum Classic; Expanse; Factom; Feathercoin; Gambit; GameCredits; Groestlcoin; HyperSpace; Ixcoin; LBRY Credits; Lisk; Litecoin; MaidSafeCoin; Maxcoin; Monero; NeosCoin; NuShares; Nxt; Nyancoin; Peercoin; Primecoin; Quark; Radium; Siacoin; Steem; Trollcoin; Ubiq; Verge; Waves; X808Coin; Zeitcoin; | Counterparty; Karbo; Neutron; Syndicate; Triangles | Aeon | Boolberry; ECC; NewYorkCoin; Piggycoin; Qwark; Universal Currency | X42 coin; PandaCoin | | |

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|---|---|---|---|---|---|---|---|---|---|---|
| | | TagCoin; Terracoin; TeslaCoin; TransferCoin; Truckcoin; TrumpCoin; UltraCoin; Unitus; Unobtanium; VeriCoin; Woodcoin; WorldCoin; Xaurum; XRP; Zetacoin; | | | | | | | | |
| Alpha-Stable distribution | Symmetric generalized hyperbolic motions | | | | | Asymmetric generalized hyperbolic motions | | | | |
| | GHYP | HYP | NIG | VG | T | GHYP | HYP | NIG | VG | T |
| *Cryptocurrencies, period 01.09.2016 – 01.06.2017 (sub-period 1)* <br> *N = 192* | | | | | | | | | | |

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|---|---|---|---|---|---|---|---|---|---|---|
| Decred; DNotes; Emercoin; Fastcoin; I0Coin; NeosCoin; Primecoin; Rise; Siacoin; Sphere; TittieCoin | X808Coin; Bitcoin_Scrypt; Capricoin; Diamond; Karbo; MaidSafeCoin; Quark; Rimbit; Viacoin; Woodcoin; Yocoin | Adzcoin; Argentum; ArtByte; AudioCoin; BitBay; Bitcoin; BitCrystals; Bitmark; BitSend; bitSilver; Bitstar; Bitswift; bitUSD; Bitzeny; BlackCoin; Blocknet; BlueCoin; Boolberry; Breakout_Stake; Bullion; BunnyCoin; Bytecoin; Canada_eCoin; CasinoCoin; Circuits_of_Value; Clams; Counterparty; Crown; Cryptonite; Curecoin; Dash; Deutsche_eMark; DigiByte; DigitalNote; DigitalPrice; DigixDAO; DopeCoin; Dotcoin; E_Dinar_Coin; e_Gulden; EDRCoin; Einsteinium; Elementrem; Energycoin; Ethereum; Ethereum_Classic; EverGreenCoin; ExclusiveCoin; Factom; FairCoin; Feathercoin; FLO; FoldingCoin; FujiCoin; GameCredits; GeoCoin; Global_Currency_Reserve; GoldCoin; GridCoin; Groestlcoin; HEAT; HiCoin; HunterCoin; HyperSpace; I_O_Coin; Kobocoin; Kore; Lisk; Litecoin; Manna; Megacoin; Memetic PepeCoin; MintCoin; MonaCoin; MonetaryUnit; Myriad; NavCoin; Nexus; NobleCoin; Novacoin; NuBits; NuShares; OBITS; OKCash; Orbitcoin; Pesetacoin; PetroDollar; PopularCoin; PutinCoin; Qwark; Ratecoin; Shift; SIBCoin; SpreadCoin; Stealth; Steem; Steem_Dollars; Stellar; Stratis; Syndicate; Terracoin; TransferCoin; Truckcoin; Ubiq; UltraCoin; Unitus; Universal_Currency; Unobtanium; Verge; X42_coin; Xaurum; | Bitcoin Fast; SaluS | Auroracoin; Dimecoin; Gambit; GlobalBoost_Y; HOdlcoin; LEOcoin; Nullex; Nyancoin; PotCoin; Radium; ReddCoin; Safe Exchange Coin; SixEleven; Syscoin; TagCoin; TeslaCoin; TrumpCoin; Waves; WhiteCoin; | bitEUR; Burst; Dogecoin; Expanse; Peercoin; PinkCoin; SolarCoin; Zetacoin | bitCNY; Freicoin; LiteDoge; Moin; Pandacoin; Rubycoin; Triangles; WorldCoin; Bitcoin Plus | Augur; Bata; GCN_Coin; Gulden; Ixcoin; Piggycoin; Pura; SmileyCoin; Zeitcoin | bitBTC | PIVX | Advanced Internet Blocks; Cannabis Coin; Chess Coin; Elcoin; Marte Xcoin; Monero; Namecoin; Neutron; VeriCoin; XRP |

| Alpha-Stable distribution | Symmetric generalized hyperbolic motions | | | | | Asymmetric generalized hyperbolic motions | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | GHYP | HYP | NIG | VG | T | GHYP | HYP | NIG | VG | T |
| *Cryptocurrencies, period 02.06.2017 – 01.03.2018 (sub-period 2)* <br> *N = 191* | | | | | | | | | | |
| Adzcoin; Bean_Cash; bitCNY; bitSilver; bitUSD; BlueCoin; BunnyCoin; Bytecoin; Capricoin; CasinoCoin; Deutsche eMark; DigiByte; DigitalNote; DigitalPrice; DigixDAO; E Dinar Coin; Elementrem; Factom; FoldingCoin; Freicoin; Global Currency Reserve; GlobalBoost_Y; Groestlcoin; HEAT; HunterCoin; HyperStake; I_O_Coin; Litecoin; | Bitcoin Scrypt; Diamond; Karbo; Quark; Rimbit; Viacoin; Woodcoin; Yocoin | Argentum; ArtByte; AudioCoin; BitBay; Bitcoin; BitCrystals; Bitmark; BitSend; Bitstar; Bitswift; Bitzeny; BlackCoin; Blocknet; Boolberry; Breakout_Stake; Bullion; Canada_eCoin; Circuits_of_Value; Clams; Counterparty; Crown; Cryptonite; Curecoin; Dash; DopeCoin; Dotcoin; e_Gulden; EDRCoin; Einsteinium; Energycoin; Ethereum; Ethereum_Classic; EverGreenCoin; ExclusiveCoin; FairCoin; Feathercoin; FLO; FujiCoin; GameCredits; GeoCoin; GoldCoin; GridCoin; HiCoin; HyperSpace; Kobocoin; Kore; Lisk; Manna; Megacoin; Memetic___PepeCoin; MonetaryUnit; Myriad; NavCoin; Nexus; NobleCoin; Novacoin; NuBits; OBITS; OKCash; Orbitcoin; Pesetacoin; PetroDollar; PopularCoin; Qwark; Shift; SpreadCoin; Stealth; Stellar; Syndicate; Terracoin; TransferCoin; Truckcoin; Ubiq; UltraCoin; Universal_Currency; Unobtanium; Verge; X42_coin; Xaurum; | Bitcoin Fast; SaluS | Auroracoin; Gambit; HOdlcoin; LEOcoin; Nullex; Nyancoin; PotCoin; Radium; Safe Exchange Coin; SixEleven; Syscoin; TagCoin; TeslaCoin; WhiteCoin | bitEUR; Burst; Dogecoin; Expanse; Peercoin; PinkCoin; SolarCoin; Zetacoin | Bitcoin Plus; LiteDoge; Triangles; WorldCoin | Augur; Bata; GCN_Coin; Gulden; Ixcoin; Piggycoin; Pura; Rise; SmileyCoin ; Sphere; Zeitcoin | bitBTC | Fast coin; PIV X | Advanced Internet Blocks; Anoncoin; Cannabis Coin; Chess Coin; Decred; Elcoin; I0Coin; Marte Xcoin; Monero; Namecoin; Neutron; Primecoin ;Startcoin; VeriCoin; XRP |

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|---|---|---|---|---|---|---|---|---|---|---|
| MaidSafeCoin; MintCoin; Moin; MonaCoin; NeosCoin; NuShares; Nxt; PutinCoin; Ratecoin; ReddCoin; Rubycoin; Siacoin; SIBCoin; Steem; Steem Dollars; Stratis; TrumpCoin; Waves; X808Coin; | | | | | | | | | | |
| Alpha-Stable distribution | Symmetric generalized hyperbolic motions | | | | | Asymmetric generalized hyperbolic motions | | | | |
| | GHYP | HYP | NIG | VG | T | GHYP | HYP | NIG | VG | T |
| *Cryptocurrencies, period 02.03.2018 – 01.02.2019 (sub-period 3)* *N = 189* | | | | | | | | | | |
| Auroracoin; BitShares; Bolivarcoin; DigiByte; Expanse; FairCoin; Memetic PepeCoin; Myriad; NavCoin; Truckcoin; Yocoin; Zeitcoin | Advanced_Internet_Blocks; Aeon; CloakCoin; DopeCoin; ECC; Ethereum; Ethereum_Classic; HiCoin; HyperStake; Litecoin; | Adzcoin; AudioCoin; Augur; AurumCoin; Bata; bitBTC; Bitcoin_Fast; Bitcoin_Plus; BitCrystals; bitEUR; bitGold; Bitmark; bitSilver; Bitswift; Bitzeny; Blocknet; Boolberry; Breakout; Breakout_Stake; Bullion; Canada_eCoin; CannabisCoin; CasinoCoin; ChessCoin; Clams; Cryptonite; Deutsche_eMark; Digitalcoin; DigitalPrice; DigixDAO; Dimecoin; Dogecoin; E_Dinar_Coin; e_Gulden; EDRCoin; Einsteinium; Elcoin; Energycoin; EverGreenCoin; ExclusiveCoin; Factom; | MonetaryUnit | Bean_Cash; BitBar; BitBay; BlueCoin; Crown; Curecoin; Decred; Elementrem; Emercoin; Gambit; GeoCoin; GlobalBoost_Y; GoldCoin; ION; MintCoin; Nexus; NobleCoin; Pandacoin; SaluS; TittieCoin; Triangles; | AsiaCoin; Circuits of Value; Creditbit; Quark | Burst; Counterparty; Megacoin | Bela; BitSend; DNotes; Moin; Namecoin; NEM; PetroDollar; Unitus | Bitcoin; Bytecoin; HunterCoin | Woodcoin; WorldCoin | Anoncoin; HempCoin; Startcoin |

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| | LiteDoge; MaidSafeCoin; Monero; Nxt; PopularCoin; ReddCoin; Sphere; Steem; | Fastcoin; Feathercoin; FLO; Freicoin; FujiCoin; GameCredits; Global_Currency_Reserve; GridCoin; Groestlcoin; Gulden; HEAT; HOdlcoin; I_O_Coin; I0Coin; Ixcoin; Karbo; Kobocoin; Kore; LEOcoin; Magi; Manna; MarteXcoin; Maxcoin; MonaCoin; NeosCoin; NewYorkCoin; Novacoin; NuBits; Nullex; NuShares; OBITS; Omni; Orbitcoin; PACcoin; PayCoin; Peercoin; Phoenixcoin; Piggycoin; PinkCoin; PIVX; PotCoin; Primecoin; PutinCoin; Radium; Ratecoin; RevolutionVR; Rise; Rubycoin; Safe Exchange Coin; SIBCoin; SixEleven; SmileyCoin; SolarCoin; SpreadCoin; Stealth; Stratis; Syndicate; Syscoin; TagCoin; Terracoin; TransferCoin; Trollcoin; TrumpCoin; Unobtanium; Waves; X808Coin; Xaurum; XRP | | UltraCoin; Vertcoin; WhiteCoin; X2GIVE; Zetacoin | | | | | | |
| *NASDAQ TOP100 stocks, period 01.09.2016 – 01.02.2019*<br>*N = 97* | | | | | | | | | | |
| AMAT; CMCSA; HSIC; INTC; KLAC; MU; PEP; XEL | ADI; ASML; LRCX; MCHP; NFLX; NXPI | AAL; AAPL; ADBE; ADP; ADSK; AMGN; AMZN; ATVI; AVGO; BIIB; CDNS; CELG; CERN; CHTR; COST; CSCO; CTRP; CTXS; EA; FB; FISV; GILD; IDXX; INCY; INTU; KHC; LBTYA; LULU; MDLZ; MSFT; MXIM; MYL; NTAP; PYPL; QCOM; REGN; ROST; SWKS; SYMC; TMUS; UAL; ULTA; VRSN; VRTX; WDAY; WDC; WLTW; WYNN; XLNX; | | ALXN; BMRN; CTAS; ISRG; JBHT; LBTYK; MAR; MNST; PAYX; PCAR; SNPS; TTWO; WBA | GOOG; GOOGL; JD; MELI; TSLA; TXN | DLTR; EBAY; NVDA | ALGN; CHKP; CSX; SIRI | EXPE | FAST | AMD; HAS; ILMN; NTES; SBUX; VRSK |

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