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Topological Methods in Economics: From Equilibrium Existence to Topological Data Analysis

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Abstract: This paper surveys how the three branches of topology — differential, algebraic, and point-set — enter mathematical economics, and organises classical and contemporary results within a single conceptual frame. The classical layer rests on four pillars: Sard's theorem and the preimage theorem, which underwrite Smale's convexity-free proof of Walrasian equilibrium existence and Debreu's local-uniqueness theorem for generic economies; the Brouwer and Kakutani fixed-point theorems behind the Arrow–Debreu existence proof; the Chichilnisky–Heal homotopy obstruction that ties continuous anonymous social choice to contractibility of preferred cones, and through it to the existence of competitive equilibrium; and the outer Hausdorff metric of Berliant and ten Raa, which makes location theory well-posed on the infinite-dimensional commodity space of land parcels. The contemporary layer turns the same apparatus toward data. The L^1 norm of H_1 persistence landscapes computed on rolling windows of asset returns rises three to eight weeks before the 2008 and 2015–2016 market crashes, outperforming the VIX in lead time; Ollivier–Ricci curvature on equity correlation networks supplies a complementary leading indicator of systemic fragility. Optimal transport recasts identifiability in matching markets as uniqueness of a dual Kantorovich plan, governed by the topology of the type-space supports. The manifold hypothesis makes nonparametric instrumental-variable convergence rates depend on intrinsic rather than ambient dimension. Together these results trace a coherent arc from fixed-point theorems to topological data analysis; a Morse-theoretic synthesis on the state manifold of an economy, linking equilibria, tâtonnement dynamics, and saddle-point crises, is stated as a research programme rather than a theorem.

Keywords: differential topology, persistent homology, topological data analysis, Walrasian equilibrium, optimal transport, social choice, Ollivier–Ricci curvature, manifold hypothesis

JEL codes: B16, C02, D50, G01, C14

1 Introduction

Topology has been a working tool of mathematical economics for almost a century, but only sometimes a conscious one. The Arrow–Debreu existence theorem (Arrow and Debreu 1954) rests on a topological fact about correspondences on a simplex; the Sonnenschein–Mantel–Debreu pathology rests on another about the range of aggregate excess demand; Chichilnisky's impossibility theorem (Chichilnisky 1980) rests on a third about the contractibility of preference spaces. None of the three was written down as a topological statement first, and each became cleaner once it was.

The claim of these notes is straightforward. Economics has used topology more often than it has studied it, and the cases where the topology was overlooked — not where it was missing — account for many of the field's classical impossibility and indeterminacy results. Recent work pushes in the opposite direction. Persistent homology of asset returns (Gidea and Katz 2018) treats a rolling window of log-returns as a point cloud and tracks how its loops and connected components evolve; the resulting persistence landscape norm rises three to eight weeks before the 2008 and 2015–2016 crashes, when the VIX shows nothing comparable. The Acemoglu–Ozdaglar–Tahbaz-Salehi analysis of contagion in interbank networks (Acemoglu et al. 2015) turns on which connected components of the liability graph survive the bankruptcy of a given node. Galichon and Salanié (Galichon and Salanié 2022) reduce identification in matching markets to uniqueness of an optimal transport plan, which in turn depends on the topology of the supports of the worker and firm distributions. And the optimal sup-norm rates of Chen and Christensen (Chen and Christensen 2018) for nonparametric IV regression are governed by the intrinsic (manifold) dimension of the parameter space, not by its ambient dimension — the manifold hypothesis of Carlsson (Carlsson 2009) made quantitative.

What links the four applications is a single shift. The early use of topology in economics asked: does an equilibrium exist? The recent use asks: what does the data look like? The same apparatus — manifolds, fixed-point theorems, homotopy, persistent homology — now reads both questions.

The three branches of topology enter at different points of the shift just described. Differential topology — manifolds, Sard's theorem, the preimage theorem — handles existence and local uniqueness of Walrasian equilibrium, the optimal transport problem behind matching, and the Bessel reparametrisation of interest-rate models (Maghsoodi 1996). Algebraic topology — Brouwer's fixed-point theorem, homotopy groups, persistent homology — handles the existence proof in its classical form, the impossibility of continuous anonymous social choice (Chichilnisky and Heal 1983), and the detection of cycles and crises in market data (Carlsson 2009; Gidea and Katz 2018). Point-set and metric topology — the outer Hausdorff metric — supplies the foundation for location theory once one tries to put a meaningful topology on the infinite-dimensional commodity space of parcels of land (Berliant & ten Raa 1988).

Differential topology carries the largest applied burden of the three; Figure 1 maps the territory.

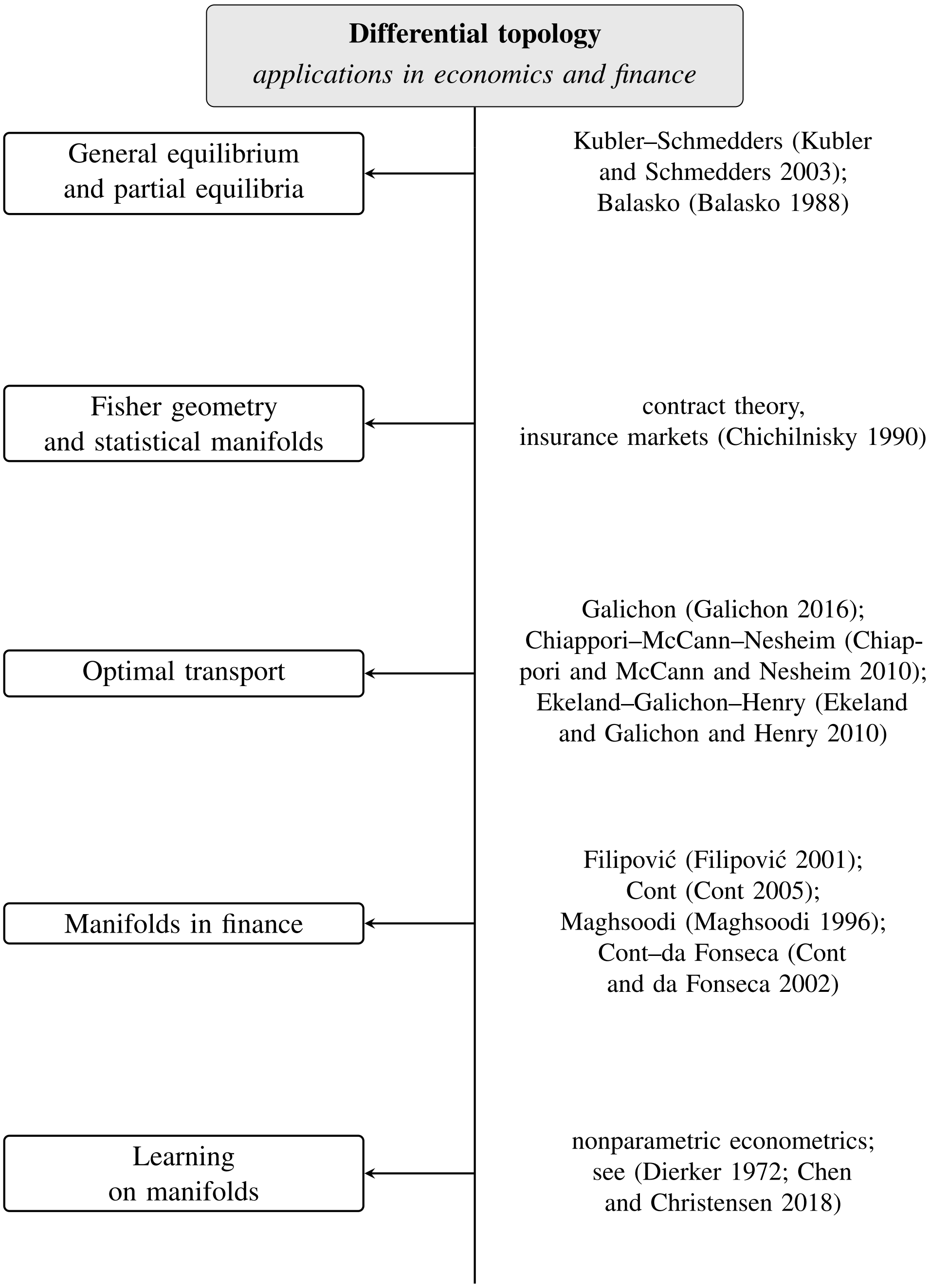


Figure 1: Main directions of applications of differential topology in economics and finance (compiled by the author from the cited literature).

The classical development below follows Milnor's *Topology from the Differentiable Viewpoint* (Milnor 1965) through the well-organised notes of Du (Du n.d.), extended where needed with results of Smale (Smale 1974), Chichilnisky (Chichilnisky 1993) and Berliant and ten Raa (Berliant & ten Raa 1988). The crisis-diagnostics section then carries the same apparatus into the world of Morse theory and persistent homology (Smale 1974; Chichilnisky 1993; Gidea and Katz 2018).

Why should an economist know topology?

Four working applications, all backed by enough applied literature and software (Ripser, Gudhi, Giotto-TDA, POT) to count as standard tools rather than curiosities, motivate the rest.

(1) Labour markets and matching: optimal transport as equilibrium. A labour market model with heterogeneous workers and firms leads to the Monge–Kantorovich equation. Galichon and Salanié (Galichon and Salanié 2022) show that identifiability of the matching surplus $\Phi(x,y)$ is equivalent to uniqueness of the dual transport problem — which in turn depends on the topology of the supports of the worker and firm distributions (cf. (Chiappori and McCann and Nesheim 2010; Galichon 2016)). This model is now the standard framework in empirical research on occupational segregation and the marriage market; the topology of the type space governs whether the main identification theorem holds.

(2) Crisis detection by persistent homology. Gidea and Katz (Gidea and Katz 2018) construct the Vietoris–Rips complex on a point cloud of log-returns of assets in a rolling time window. The barcodes of the homology group H_1 exhibit a significant increase in the persistence landscape norm $L_1(t)$ 3–8 weeks before the 2008 and 2015–2016 crashes, without a comparable signal from the VIX. This is a direct realisation of persistent homology from Section 11: growing H_1 barcodes correspond to approach towards a saddle point of the Morse function $V : \mathscr{S} \to \mathbb{R}$ (Theorem 11.6).

(3) High-dimensional econometrics and the manifold hypothesis. High-dimensional economic data concentrate on a low-dimensional manifold embedded in a large feature space (*manifold hypothesis*). Efficient estimation consists in discovering this manifold; sup-norm convergence rates depend on the manifold's intrinsic dimension, not on the ambient dimension — as formalised by Chen and Christensen (Chen and Christensen 2018). The topological foundation — how many connected components, loops and higher holes the manifold has — is now addressed by TDA (Carlsson 2009).

(4) Stability of financial networks. Acemoglu, Ozdaglar and Tahbaz-Salehi (Acemoglu et al. 2015) show that the phase transition between the diversification regime and the systemic contagion regime is topological: it occurs when the connected components of the liability network change structure discontinuously. This is a concrete instance of a change in Morse index (Proposition 11.3) in the state space of the financial system.

Sard's theorem and the preimage theorem, homotopy groups, the Hausdorff metric, persistent homology — these are the tools behind all four examples, and Figure 2 traces the path from each theorem to where it does its empirical work.

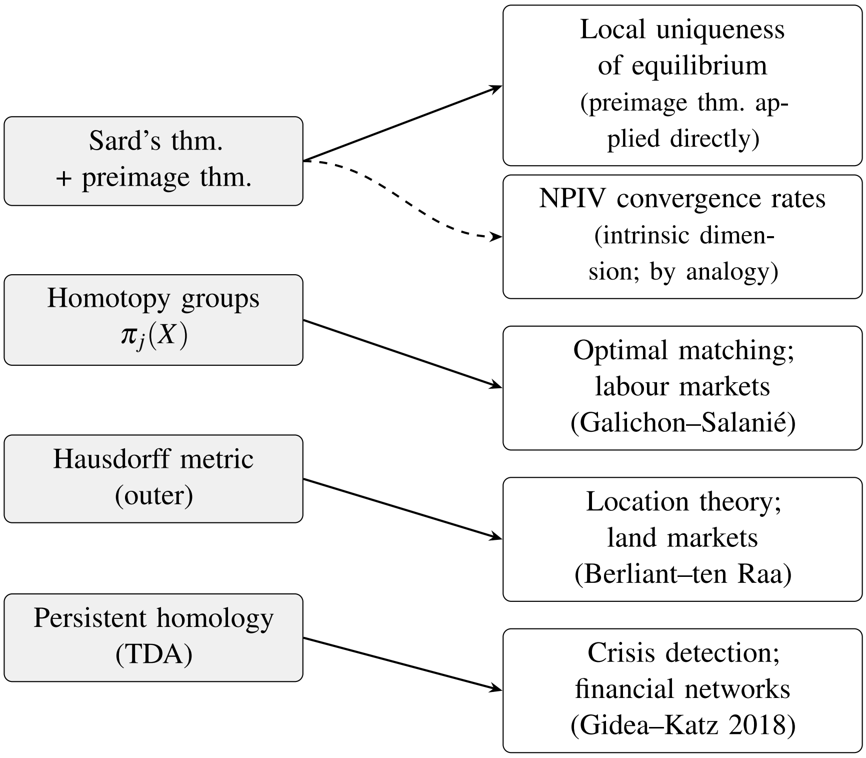


Figure 2: From abstract topological theorems to contemporary empirical applications in economics. Solid arrows denote direct logical dependence; the dashed arrow indicates an application by analogy (intrinsic-dimension argument in the spirit of the preimage theorem, without full manifold structure).

2 Smooth Manifolds and Derivatives

The four working applications in the introduction all live on smooth manifolds: the price simplex for general equilibrium, the type space for matching, the parameter manifold for nonparametric instrumental variables, the state manifold for crisis detection. Before any of them can be made precise, the underlying smooth structure needs spelling out: what counts as smooth, what counts as a manifold, and what counts as a derivative between them. That is the content of this section.

2.1 Smooth maps

Definition 2.1 (Smooth map, I)**.** *Let U be an open subset of $\mathbb{R}^k$ and Y any subset of $\mathbb{R}^l$. A map $f\colon U \to Y$ is* smooth *if at every point of U all partial derivatives of every order exist and are continuous.*

Definition 2.2 (Smooth map, II)**.** *Let $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^l$. A map $f\colon X \to Y$ is* smooth *if for every $x \in X$ there exists an open set $U \subset \mathbb{R}^k$ containing x and a smooth extension $F\colon U \to \mathbb{R}^l$ (in the sense of the previous definition) that agrees with f on $U \cap X$.*

Smoothness is preserved under composition:

Proposition 2.3 (Du (Du n.d.), Prop. 2.1.1)**.** *Let $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^l$, $Z \subset \mathbb{R}^j$, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be smooth. Then $g \circ f\colon X \to Z$ is smooth.*

Proof. Fix $x \in X$ and let $y = f(x) \in Y$. Since f is smooth at x, there exists an open $U \subset \mathbb{R}^k$ containing x and a smooth extension $F\colon U \to \mathbb{R}^l$ with $F|_{U\cap X} = f|_{U\cap X}$. Since g is smooth at y, there exists an open $V \subset \mathbb{R}^l$ containing y and a smooth extension $G\colon V \to \mathbb{R}^j$ with $G|_{V\cap Y} = g|_{V\cap Y}$. Set $U_0 = F^{-1}(V) \subset U$; as the preimage of an open set under the continuous map F, the set U_0 is open in $\mathbb{R}^k$ and contains x. The composition $G \circ F|_{U_0}\colon U_0 \to \mathbb{R}^j$ is smooth as a composition of smooth maps between open subsets of Euclidean spaces. For every $z \in U_0 \cap X$ we have $(G \circ F)(z) = G(f(z)) = (g \circ f)(z)$, so $G \circ F|_{U_0}$ is a smooth extension of $g \circ f$ at x. □

Definition 2.4 (Diffeomorphism)**.** *A map $f\colon X \to Y$ is a* diffeomorphism *if it is a bijection and both f and f^{-1} are smooth.*

Definition 2.5 (Smooth manifold)**.** *A subset $M \subset \mathbb{R}^k$ is a* smooth manifold of dimension m *if every point $x \in M$ has a neighbourhood U in M diffeomorphic to an open subset $\varphi(U) \subset \mathbb{R}^m$. The map $\varphi\colon U \to \varphi(U)$ is called a* local chart, *and the family of all charts covering M is an* atlas.

The following figure illustrates the standard convention used in the literature (cf. Milnor (Milnor 1965), Lee (Lee 2003)): an irregular patch M, a distinguished open subset $U \subset M$ around a point x, and a local chart $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^m$ "flattening" the neighbourhood of x onto a Euclidean domain.

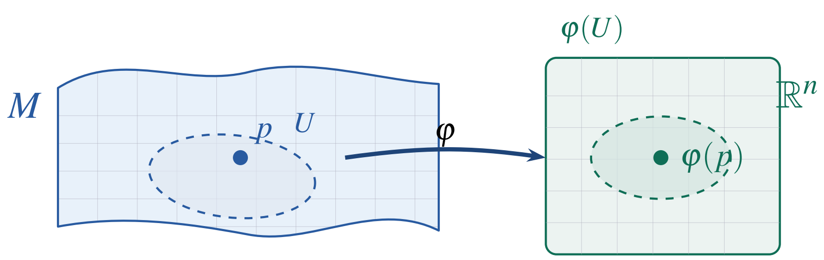


Figure 3: Local chart $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^n$ on a smooth manifold M.

2.2 Tangent spaces

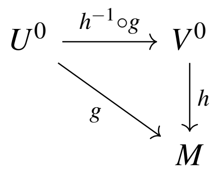
Definition 2.6 (Tangent space)**.** *For a smooth m-dimensional manifold $M \subset \mathbb{R}^k$ and a point $x \in M$, choose a parametrisation $g\colon U \to M$ of a neighbourhood $g(U) \ni x$, where U is open in $\mathbb{R}^m$. The* tangent space *of M at x is*

$$T_x(M) = dg_u(\mathbb{R}^m), \quad g(u) = x,$$

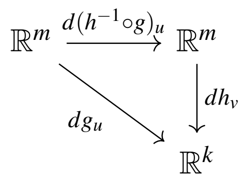
where the derivative dg_u is in the sense of Definition 2.2.2 of Du (Du n.d.).

Proposition 2.7 (Du (Du n.d.), Prop. 2.2.4)**.** *$T_x(M)$ is independent of the choice of parametrisation.*

Proof. Let $h\colon V \to M$ be a second parametrisation of a neighbourhood of x, $v = h^{-1}(x)$. Set $W = g(U) \cap h(V)$, $U^0 = g^{-1}(W)$, $V^0 = h^{-1}(W)$. Then $h^{-1} \circ g\colon U^0 \to V^0$ is a diffeomorphism. We have the following commutative diagram (Du (Du n.d.), p. 4):



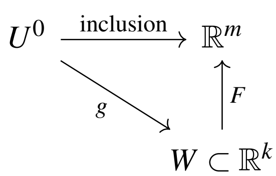
Taking derivatives and applying the chain rule:



Since $h^{-1} \circ g$ is a diffeomorphism, $d(h^{-1} \circ g)_u$ is an isomorphism. From the diagram we conclude $dg_u(\mathbb{R}^m) = dh_v(\mathbb{R}^m)$. □

Proposition 2.8 (Du (Du n.d.), Prop. 2.2.5)**.** *$T_x(M)$ is an m-dimensional subspace of $\mathbb{R}^k$.*

Proof. Let $g^{-1}\colon g(U) \to U$ be a smooth map; let $F\colon W \to \mathbb{R}^m$ (W open in $\mathbb{R}^k$) be its smooth extension, and $U^0 = g^{-1}(W \cap g(U))$. Diagram (Du (Du n.d.), p. 4):



Taking derivatives:

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{\ \mathrm{id}\ } & \mathbb{R}^m \\ & \searrow^{dg_u} & \uparrow dF_x \\ & & \mathbb{R}^k \end{array}$$

The diagram yields injectivity of dg_u, hence $\dim T_x(M) = m$. □

2.3 The derivative between manifolds

Definition 2.9 (Derivative; Du (Du n.d.), Def. 2.2.4)**.** *For smooth $f\colon M \to N$, $x \in M$, $y = f(x)$, with smooth extension $F\colon W \to \mathbb{R}^l$ ($W \ni x$ open in $\mathbb{R}^k$):*

$$df_x(v) = dF_x(v), \quad v \in T_x(M).$$

Proposition 2.10 (Du (Du n.d.), Prop. 2.2.6)**.** *$df_x\colon T_x(M) \to T_y(N)$ is well-defined and independent of the extension F.*

Proof. Choose parametrisations $g\colon U \to M$ and $h\colon V \to N$ ($g(u) = x$, $h(v) = y$) such that $f(g(U)) \subset h(V)$. Diagram (Du (Du n.d.), p. 5):

$$\begin{array}{ccc} W & \xrightarrow{\ F\ } & \mathbb{R}^l \\ g \uparrow & & \uparrow h \\ U & \xrightarrow[h^{-1}\circ f\circ g]{} & V \end{array}$$

Taking derivatives:

$$\begin{array}{ccc} \mathbb{R}^k & \xrightarrow{\ dF_x\ } & \mathbb{R}^l \\ dg_u \uparrow & & \uparrow dh_v \\ \mathbb{R}^m & \xrightarrow[d(h^{-1}\circ f\circ g)_u]{} & \mathbb{R}^n \end{array}$$

The diagram gives $dF_x \circ dg_u = dh_v \circ d(h^{-1} \circ f \circ g)_u$, so dF_x maps $T_x(M) = dg_u(\mathbb{R}^m)$ into $T_y(N) = dh_v(\mathbb{R}^n)$. Since dg_u is injective (Proposition 2.8), the value $dF_x|_{T_x(M)}$ is uniquely determined by the right-hand side, which is independent of F; thus df_x is independent of the extension F. □

Theorem 2.11 (Inverse function theorem)**.** *If $f\colon U \to \mathbb{R}^k$ is smooth, $U \subset \mathbb{R}^k$ open, and df_{x_0} is non-singular, then f maps some neighbourhood of x_0 diffeomorphically onto an open set $f(U_0) \subset \mathbb{R}^k$.*

Theorem 2.12 (Preimage theorem; Du (Du n.d.), Thm. 2.3.4)**.** *Let $f\colon M \to N$ be smooth,* $\dim M = m \geq n = \dim N$, *and $y \in N$ a regular value with $f^{-1}(y) \neq \emptyset$. Then $f^{-1}(y)$ is a smooth manifold of dimension $m - n$.*

3 Sard's Theorem

Smooth manifolds and derivatives between them, the subject of the previous section, raise a question that does not arise for general continuous maps: when a smooth $f\colon M \to N$ is at full rank, the preimage theorem tells us that $f^{-1}(y)$ is itself a smooth manifold of dimension $\dim M - \dim N$. The question is how exceptional the failure of full rank really is. Sard's theorem gives the answer: the set of points where the rank collapses, measured in the target, has Lebesgue measure zero. The local uniqueness of Walrasian equilibria in Section 6 is a direct corollary, and the same genericity argument reappears in the crisis-detection section through Morse functions.

3.1 Statement

Definition 3.1 (Critical points and critical values)**.** *A point $x \in M$ is* critical *for $f\colon M \to N$ if* $\operatorname{rank}(df_x) < \dim N$. *The set $C(f)$ denotes the critical points; $f(C(f))$ the critical values.*

Theorem 3.2 (Sard, 1942; Du (Du n.d.), Thm. 2.3.1 and 2.3.3)**.** *The set of critical values of a smooth map $f\colon M \to N$ has Lebesgue measure zero in N.*

Proof. Following Du (Du n.d.), §2.4 (after Milnor (Milnor 1965), §3).

It suffices to prove the result locally on coordinate charts, so we may assume $f\colon U \to \mathbb{R}^n$ where $U \subset \mathbb{R}^m$ is open. Write $C \subset U$ for the set of critical points of f, and let

$$C_i = \{x \in U \mid \text{all partial derivatives of } f \text{ of order } \le i \text{ vanish at } x\}, \qquad i \ge 1.$$

By definition $C \supset C_1 \supset C_2 \supset \cdots$. The argument proceeds by induction on m.

Base case ($m = 0$). Then U is a single point and $f(U)$ is a single point in $\mathbb{R}^n$, which has measure zero (assuming $n \ge 1$; for $n = 0$ the statement is vacuous).

Inductive step. Suppose the theorem holds for all smooth maps from open subsets of $\mathbb{R}^{m-1}$. We show that $f(C)$ has measure zero in $\mathbb{R}^n$ by decomposing

$$f(C) = f(C \setminus C_1) \cup f(C_1 \setminus C_2) \cup \cdots \cup f(C_{k-1} \setminus C_k) \cup f(C_k),$$

with k chosen later (any $k > m/n - 1$ will do), and showing each piece has measure zero.

Step 1: $f(C \setminus C_1)$ has measure zero. Fix $\bar{x} \in C \setminus C_1$. By definition there is at least one partial derivative $\partial f_r / \partial x_j$ that does not vanish at $\bar{x}$; after permuting the coordinates of $\mathbb{R}^m$ and $\mathbb{R}^n$ we may assume $r = j = 1$, i.e. $\partial f_1 / \partial x_1(\bar{x}) \neq 0$. Define

$$h\colon \mathbb{R}^m \to \mathbb{R}^m, \qquad h(x) = (f_1(x), x_2, x_3, \ldots, x_m).$$

The Jacobian of h at $\bar{x}$ has the block form

$$dh_{\bar{x}} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\bar{x}) & * \\ 0 & I_{m-1} \end{pmatrix},$$

whose determinant equals $\partial f_1/\partial x_1(\bar{x}) \neq 0$. By the inverse function theorem (Theorem 2.11) there is an open neighbourhood $V \subset U$ of $\bar{x}$ such that $h|_V$ is a diffeomorphism onto $h(V) \subset \mathbb{R}^m$.

Consider the smooth map

$$g = f \circ h^{-1} \colon h(V) \to \mathbb{R}^n.$$

Then for $y = h(x) = (f_1(x), x_2, \ldots, x_m)$ we have $g(y) = f(x)$. Crucially, the first coordinate of $g(y)$ equals $f_1(x) = y_1$, i.e.

$$g_1(y_1, y_2, \ldots, y_m) = y_1.$$

This means that g sends each horizontal slice $\{y_1 = t\} \cap h(V) \subset \mathbb{R} \times \mathbb{R}^{m-1}$ into the corresponding slice $\{z_1 = t\} \subset \mathbb{R} \times \mathbb{R}^{n-1}$. Writing $g^t \colon (\{t\} \times \mathbb{R}^{m-1}) \cap h(V) \to \{t\} \times \mathbb{R}^{n-1}$ for this restriction (a smooth map between open subsets of $\mathbb{R}^{m-1}$ and $\mathbb{R}^{n-1}$ once we drop the first coordinate), a point $y \in (\{t\} \times \mathbb{R}^{m-1}) \cap h(V)$ is critical for g^t *if and only if* it is critical for g (because $\partial g_1/\partial y_1 = 1 \neq 0$ contributes the full rank in the first row, so criticality of g depends only on the submatrix corresponding to g^t). Hence $h(C \setminus C_1) \cap V$ projects into the critical set of g^t slice by slice.

By the inductive hypothesis applied to g^t (a smooth map between open subsets of $\mathbb{R}^{m-1}$), the set of critical values of g^t has $(n-1)$-dimensional Lebesgue measure zero in $\{t\} \times \mathbb{R}^{n-1} \cong \mathbb{R}^{n-1}$, for *every* $t \in \mathbb{R}$. By Fubini's theorem

$$\text{meas}_n\big(g(\text{critical set of } g \text{ in } h(V))\big) = \int_{\mathbb{R}} \text{meas}_{n-1}\big(\text{critical values of } g^t\big)\, dt = \int_{\mathbb{R}} 0\, dt = 0.$$

Since $f = g \circ h$ and h is a diffeomorphism (so $f(C \setminus C_1) \cap V$ corresponds to the critical values of g in $h(V)$), this shows $f(C \setminus C_1) \cap V$ has measure zero in $\mathbb{R}^n$. A countable cover of $C \setminus C_1$ by such neighbourhoods V completes the step.

Step 2: $f(C_i \setminus C_{i+1})$ has measure zero for $1 \leq i < k$. Fix $\bar{x} \in C_i \setminus C_{i+1}$. By definition all partial derivatives of f of order $\leq i$ vanish at $\bar{x}$, but at least one derivative of order $i+1$ does not. That is, there exist indices r and $s_1, \ldots, s_{i+1}$ such that

$$w(x) := \frac{\partial^i f_r}{\partial x_{s_2} \cdots \partial x_{s_{i+1}}}$$

satisfies $w(\bar{x}) = 0$ (since $\bar{x} \in C_i$) but $\partial w/\partial x_{s_1}(\bar{x}) \neq 0$ (since $\bar{x} \notin C_{i+1}$). After relabelling we may take $s_1 = 1$, so $\partial w/\partial x_1(\bar{x}) \neq 0$.

Now define

$$h\colon \mathbb{R}^m \to \mathbb{R}^m, \qquad h(x) = \big(w(x), x_2, x_3, \ldots, x_m\big).$$

By the same Jacobian computation as in Step 1, $\det dh_{\bar{x}} = \partial w/\partial x_1(\bar{x}) \neq 0$, so h is a local diffeomorphism on some neighbourhood V of $\bar{x}$.

The key observation is that w vanishes identically on C_i (because all i-th derivatives of f vanish on C_i), so

$$h(C_i \cap V) \subset \{0\} \times \mathbb{R}^{m-1}.$$

That is, h carries $C_i \cap V$ into a hyperplane of one lower dimension. Consider the restricted map

$$\widetilde{f} = f \circ h^{-1}\colon\ h(V) \to \mathbb{R}^n,$$

and its further restriction to the hyperplane:

$$\widetilde{f}|_{(\{0\}\times\mathbb{R}^{m-1})\cap h(V)}\colon\ (\{0\} \times \mathbb{R}^{m-1}) \cap h(V) \to \mathbb{R}^n.$$

This is a smooth map from (an open subset of) $\mathbb{R}^{m-1}$ to $\mathbb{R}^n$. Every point of $h(C_i \cap V)$ lies in its domain.

By the inductive hypothesis (applied to this map from $\mathbb{R}^{m-1}$ to $\mathbb{R}^n$), the image $\widetilde{f}\big((\{0\} \times \mathbb{R}^{m-1}) \cap h(V)\big)$ has measure zero in $\mathbb{R}^n$. In particular its subset $f(C_i \cap V) \supset f((C_i \setminus C_{i+1}) \cap V)$ has measure zero. Countable cover of $C_i \setminus C_{i+1}$ by such V finishes Step 2.

Step 3: $f(C_k)$ has measure zero for $k > m/n - 1$. Cover the closure of U by countably many compact cubes; it suffices to bound $\operatorname{meas}_n(f(C_k \cap I^m))$ for a single closed cube $I^m \subset U$ of side length $\delta > 0$.

By Taylor's theorem applied at each $x \in C_k \cap I^m$, all partial derivatives of f of order $\leq k$ vanish at x, so for every $h \in \mathbb{R}^m$ with $x + h \in I^m$,

$$f(x+h) = f(x) + R(x,h), \qquad \|R(x,h)\| \leq c\,\|h\|^{k+1},$$

where c is a uniform bound (depending on f and I^m) on the $(k+1)$-st derivatives of f over I^m. This c is finite because f is smooth on a compact set.

Subdivide I^m into r^m closed subcubes of side δ/r, for an integer $r \geq 1$ to be chosen. Let I^m_ν be such a subcube containing some point $x_\nu \in C_k \cap I^m_\nu$ (if no such point exists, ignore the cube). For any $x \in I^m_\nu$ we have $\|x - x_\nu\| \leq \sqrt{m}\,\delta/r$ (diameter of the cube), and so

$$\|f(x) - f(x_\nu)\| \leq c\,(\sqrt{m}\,\delta/r)^{k+1} = \frac{c\,(\sqrt{m}\,\delta)^{k+1}}{r^{k+1}} =: \frac{a}{r^{k+1}},$$

where $a = c\,(\sqrt{m}\,\delta)^{k+1}$ depends only on f and I^m.

Hence $f(I^m_\nu)$ is contained in a closed ball of radius a/r^{k+1} in $\mathbb{R}^n$, whose Lebesgue measure is bounded by

$$\operatorname{meas}_n\big(f(I^m_\nu)\big) \le \big(2a/r^{k+1}\big)^n = (2a)^n \cdot r^{-(k+1)n}.$$

Summing over the at-most-r^m relevant subcubes:

$$\operatorname{meas}_n\big(f(C_k \cap I^m)\big) \le r^m \cdot (2a)^n \cdot r^{-(k+1)n} = (2a)^n \cdot r^{m-(k+1)n}.$$

Choose k so that $m-(k+1)n < 0$, i.e. $k > m/n - 1$. Then $r^{m-(k+1)n} \to 0$ as $r \to \infty$, so $\operatorname{meas}_n(f(C_k \cap I^m)) = 0$. This completes Step 3.

Combining Steps 1–3 gives $\operatorname{meas}_n(f(C)) = 0$. □

Corollary 3.3. *For almost every $y \in N$, $f^{-1}(y)$ is a smooth manifold of dimension* $\dim M - \dim N$ *(when non-empty).*

Proof. By Sard's theorem, the set of critical values of f has Lebesgue measure zero in N. Therefore every $y \in N$ outside this null set is a regular value of f. If $f^{-1}(y) = \varnothing$, there is nothing to prove. If $f^{-1}(y) \neq \varnothing$, the preimage theorem (Theorem 2.12) applies and shows that $f^{-1}(y)$ is a smooth manifold of dimension $\dim M - \dim N$. □

4 A Model of Pure Exchange

Up to this point the apparatus has been geometric and the applications notional. This section installs a working economy on which the next three sections do something concrete: prove that equilibria exist, count them, and decide when they vary smoothly with the data. Think of consumers in a city trading two or three goods — bread, milk, coffee — each with a private endowment and a smooth demand that responds to prices. The question is whether the prices ever settle.

We consider an economy with I consumers and L consumption goods. The commodity-price space is $\mathbb{R}^L_+$; endowments $\omega = (\omega^1, \ldots, \omega^I) \in \mathbb{R}^{IL}_+$. Demand function:

$$f^i : \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}^L_+.$$

Definition 4.1 (Walrasian general equilibrium; Du (Du n.d.), Def. 3.1.1)**.** *The economy is in* Walrasian equilibrium *if*

$$\sum_{i=1}^{I} f^i(p, p \cdot \omega^i) = \sum_{i=1}^{I} \omega^i.$$

Definition 4.2 (Aggregate excess demand)**.**

$$Z(p,\omega) = \sum_{i=1}^{I} \big(f^i(p, p\cdot\omega^i) - \omega^i\big).$$

The economy is in equilibrium if and only if $Z(p,\omega) = 0$*.*

Assumption 4.3 (Du (Du n.d.), Assumption 3.1)**.** *Each* $f^i : \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}^L_+$ *satisfies: (1) continuity; (2)* $f^i(cp, cw) = f^i(p,w)$ *for all* $c > 0$ *(homogeneity of degree zero); (3)* $f^i(p,w)\cdot p = w$ *(Walras's law, individual). Additionally, the* aggregate *excess demand* $Z(p,\omega)$ *satisfies the following boundary condition: (4)* $p_\ell = 0 \Rightarrow Z_\ell(p,\omega) > 0$ *for all* ℓ *(free goods are in aggregate excess demand).*

Remark 4.4. *Condition (4) is a property of the aggregate function Z, not of the individual demand functions* f^i*; it is therefore of a different logical character from conditions (1)–(3). It is implied by the stronger individual-level requirement that for every consumer i and every good* ℓ*,* $f^i_\ell(p, p\cdot\omega^i) \to +\infty$ *as* $p_\ell \to 0$ *(with p ranging over* S^{L-1}_{++} *and* $p\cdot\omega^i$ *bounded away from zero). Under that individual condition, the aggregate* $Z_\ell = \sum_i f^i_\ell - \sum_i \omega^i_\ell$ *diverges to* $+\infty$ *while the endowment aggregate* $\sum_i \omega^i_\ell$ *remains finite, so (4) holds in the limit; see Mas-Colell, Whinston and Green (Mas-Colell and Whinston and Green 1995), Chapter 17. Condition (4) is thus stated at the aggregate level as a convenient shorthand, but its logical status is that of a derived property rather than an independent postulate alongside (1)–(3).*

Lemma 4.5 (Aggregate Walras's law)**.** $p\cdot Z(p,\omega) = 0$ *for all* (p,ω)*.*

Proof. For each consumer i, individual Walras's law in Assumption 4.3(3) gives

$$p\cdot f^i(p, p\cdot\omega^i) = p\cdot\omega^i.$$

Using the definition of aggregate excess demand, $Z(p,\omega) = \sum_{i=1}^{I}\big(f^i(p,p\cdot\omega^i) - \omega^i\big)$, we compute

$$p\cdot Z(p,\omega) = \sum_{i=1}^{I}\big(p\cdot f^i(p,p\cdot\omega^i) - p\cdot\omega^i\big) = \sum_{i=1}^{I}\big(p\cdot\omega^i - p\cdot\omega^i\big) = 0.$$

□

Lemma 4.6 (Du (Du n.d.), Lem. 3.2.1)**.** $Z(p,\omega) \le 0 \Rightarrow Z(p,\omega) = 0$*.*

Proof. We argue by contradiction. Suppose $Z(p,\omega) \le 0$ component-wise but $Z(p,\omega) \ne 0$, so there exists an index ℓ with $Z_\ell(p,\omega) < 0$. We show this contradicts the standing assumptions.

Since $p \in S^{L-1}_+ \subset \mathbb{R}^L_+$, every coordinate $p_{\ell'} \ge 0$, and since $Z_{\ell'}(p,\omega) \le 0$ by hypothesis, every product $p_{\ell'}Z_{\ell'}(p,\omega) \le 0$. Aggregate Walras's law gives

$$0 \;=\; p\cdot Z(p,\omega) \;=\; \sum_{\ell'=1}^{L} p_{\ell'} Z_{\ell'}(p,\omega).$$

A sum of non-positive real numbers equals zero only when every summand is zero: $p_{\ell'} Z_{\ell'}(p,\omega) = 0$ for all ℓ'.

In particular, for the index ℓ at which $Z_\ell(p,\omega) < 0$ we have $p_\ell Z_\ell(p,\omega) = 0$ with $Z_\ell(p,\omega) \neq 0$, so $p_\ell = 0$. But Assumption 4.3(4) states that $p_\ell = 0$ implies $Z_\ell(p,\omega) > 0$, contradicting $Z_\ell(p,\omega) < 0$.

Therefore no such ℓ exists, and $Z(p,\omega) = 0$. □

5 Existence of General Equilibrium

The model of the previous section is well-defined for any price vector, but it gives no reason to believe a price vector clearing the markets exists at all. The classical answer is a fixed-point argument: rescale aggregate excess demand into a continuous self-map of the price simplex, and Brouwer's theorem produces a fixed point that turns out to be the equilibrium. This is the proof of Du (Du n.d.) in the form we use, with the continuity assumption on the closed simplex made explicit so that the argument does not silently fail at the boundary.

Theorem 5.1 (Brouwer, 1910; cf. Du (Du n.d.), Thm. 3.2.3)**.** *Every continuous map $g\colon \Delta \to \Delta$ on a compact convex set $\Delta \subset \mathbb{R}^n$ has a fixed point.*

Remark 5.2 (Brouwer vs. Kakutani)**.** *The original Arrow–Debreu proof (Arrow and Debreu 1954) uses the* Kakutani fixed-point theorem *(Kakutani 1941): every upper-hemicontinuous correspondence with non-empty convex values on a compact convex set has a fixed point. Kakutani's theorem is necessary there because, when a consumer faces a budget set on which the utility maximum is not unique (e.g. with linear indifference curves), the demand $f^i(p,w)$ is a* set-valued *correspondence rather than a function, and Brouwer's theorem does not apply.*

Under Assumption 6.1 — strict convexity and smoothness of preferences — each f^i is single-valued and continuous, so the aggregate excess demand Z is a continuous function and Brouwer's theorem suffices. The use of Brouwer rather than Kakutani here is therefore a consequence of the stronger regularity imposed on individual demand, not an oversight; readers familiar with Arrow–Debreu should note this distinction.

Theorem 5.3 (Existence of equilibrium; Brouwer version)**.** *Assume Assumption 4.3. In addition, assume that for the fixed endowment vector ω the aggregate excess demand function $Z(\cdot,\omega)$ is finite and continuous on the closed price simplex S_+^{L-1}. Then there exists $p^* \in S_{++}^{L-1}$ such that $Z(p^*,\omega) = 0$.*

Proof. Define $g\colon S_+^{L-1} \to S_+^{L-1}$ by

$$g_\ell(p) = \frac{p_\ell + \max\{0, Z_\ell(p,\omega)\}}{1 + \sum_{j=1}^{L} \max\{0, Z_j(p,\omega)\}}, \qquad \ell = 1,\dots,L.$$

We first verify carefully that g is a well-defined continuous self-map of the closed simplex.

Let

$$A(p) := \sum_{j=1}^{L} \max\{0, Z_j(p,\omega)\}.$$

Because $Z(\cdot,\omega)$ is finite and continuous on S_+^{L-1} and $x \mapsto \max\{0,x\}$ is continuous on $\mathbb{R}$, the function A is continuous and finite. Moreover $A(p) \geq 0$, so $1+A(p) > 0$. Thus the denominator never vanishes, and each component $g_\ell(p)$ is continuous.

Next, $g_\ell(p) \geq 0$ because both p_ℓ and $\max\{0, Z_\ell(p,\omega)\}$ are non-negative. Finally,

$$\sum_{\ell=1}^{L} g_\ell(p) = \frac{\sum_{\ell=1}^{L} p_\ell + \sum_{\ell=1}^{L} \max\{0, Z_\ell(p,\omega)\}}{1+A(p)} = \frac{1+A(p)}{1+A(p)} = 1,$$

because $\sum_\ell p_\ell = 1$ for $p \in S_+^{L-1}$. Hence $g(p) \in S_+^{L-1}$ for every $p \in S_+^{L-1}$.

The simplex S_+^{L-1} is non-empty, compact, and convex. By Brouwer's fixed-point theorem (Theorem 5.1), there exists $p^* \in S_+^{L-1}$ such that $g(p^*) = p^*$. Put

$$A^* := A(p^*) = \sum_{j=1}^{L} \max\{0, Z_j(p^*,\omega)\}.$$

The fixed-point equation $g_\ell(p^*) = p_\ell^*$ gives

$$p_\ell^* = \frac{p_\ell^* + \max\{0, Z_\ell(p^*,\omega)\}}{1+A^*}.$$

Multiplying by $1+A^*$ and subtracting p_ℓ^* from both sides yields

$$p_\ell^* A^* = \max\{0, Z_\ell(p^*,\omega)\}. \tag{$*$}$$

Now multiply $(*)$ by $Z_\ell(p^*,\omega)$ and sum over ℓ:

$$\sum_{\ell=1}^{L} Z_\ell(p^*,\omega)\max\{0, Z_\ell(p^*,\omega)\} = A^* \sum_{\ell=1}^{L} p_\ell^* Z_\ell(p^*,\omega) = A^* p^* \cdot Z(p^*,\omega) = 0,$$

where the last equality is aggregate Walras's law. For every real number a,

$$a \max\{0,a\} = (\max\{0,a\})^2.$$

Therefore

$$\sum_{\ell=1}^{L} \bigl(\max\{0, Z_\ell(p^*,\omega)\}\bigr)^2 = 0.$$

A sum of non-negative real numbers can equal zero only if every term is zero. Hence $\max\{0, Z_\ell(p^*, \omega)\} = 0$ for every ℓ, which means $Z_\ell(p^*, \omega) \leq 0$ for every ℓ. By Lemma 4.6, this component-wise inequality implies $Z(p^*, \omega) = 0$.

It remains to prove that the equilibrium price vector is strictly positive. Suppose $p^*_\ell = 0$ for some ℓ. By Assumption 4.3(4), $Z_\ell(p^*, \omega) > 0$, contradicting the already established equality $Z(p^*, \omega) = 0$. Hence $p^*_\ell > 0$ for all ℓ, and therefore $p^* \in S^{L-1}_{++}$. □

Remark 5.4 (Continuity of g on the boundary)**.** *The opening sentence of the proof asserts that g is continuous on all of S^{L-1}_{+}, including boundary points where $p_\ell = 0$ for some ℓ. This requires that $Z(p, \omega)$ be finite at such points, which is not guaranteed by Assumption 4.3(1) alone. Indeed, for many standard demand systems derived from regular preferences, $f^i_\ell(p, p \cdot \omega^i) \to +\infty$ as $p_\ell \to 0$, so $Z_\ell(p, \omega) \to +\infty$ along any sequence approaching the boundary face $\{p_\ell = 0\}$; in that case g is not continuous on S^{L-1}_{+} and Brouwer's theorem cannot be applied directly.*

There are two standard resolutions.

(i) Truncated demand / interior approach. *Replace each f^i by its truncation $f^i_N = \min(f^i, N\mathbf{1})$ for a large N, or work on the ε-interior $S^{L-1}_{+,\varepsilon} = \{p \in S^{L-1}_{+} : p_\ell \geq \varepsilon\}$ (which is compact and convex) where Z is bounded and continuous. A fixed point of the truncated map converges to an equilibrium as $N \to \infty$ (or $\varepsilon \to 0$); the boundary condition Assumption 4.3(4) then ensures the limit lies in S^{L-1}_{++}. This is essentially the argument in Arrow and Debreu (Arrow and Debreu 1954).*

(ii) Regular preferences (Debreu's assumption). *If each preference relation $\succsim_i$ satisfies Debreu's regularity condition — the demand $f^i(p, w)$ is bounded whenever p remains in a compact subset of $\mathbb{R}^L_{+} \setminus \{0\}$ and w is bounded away from zero — then Z extends continuously to S^{L-1}_{+}, and the proof proceeds as written; see Debreu (Debreu 1970), pp. 387–388.*

In either case the conclusion of Theorem 5.3 stands, but the continuity of g on the closed simplex is a non-trivial hypothesis that depends on the demand specification, not an automatic consequence of Assumption 4.3(1).

6 Local Uniqueness of Equilibria

Existence is half the question. Once one knows an equilibrium exists, it matters whether there is one or a continuum, and whether small changes in endowments move the equilibrium continuously or jump it discontinuously between branches. Sard's theorem, prepared two sections back, now does its work: for almost every endowment vector, the set of equilibria is discrete, so equilibrium prices respond locally smoothly to the data. The proof is a dimension count plus an explicit identification of the equilibrium set with the fibre of a smooth map.

Assumption 6.1 (Du (Du n.d.), Assumption 3.2)**.** *Each* $f^i\colon \mathbb{R}^L_{++}\times\mathbb{R}_{++}\to\mathbb{R}^L_+$ *is smooth, homogeneous of degree* 0*, and satisfies Walras's law.*

Definition 6.2 (Equilibrium set; Du (Du n.d.), Def. 3.3.1)**.** $E(\omega)=\{p\in S^{L-1}_{++}\mid Z(p,\omega)=0\}$.

Define the smooth map $F\colon S^{L-1}_{++}\times\mathbb{R}_{++}\times\mathbb{R}^{(I-1)L}_{++}\to\mathbb{R}^{IL}$:

$$F(p,w^1,\omega^2,\dots,\omega^I)=\Big(f^1(p,w^1)+\sum_{i=2}^{I}f^i(p,p\cdot\omega^i)-\sum_{i=2}^{I}\omega^i,\ \omega^2,\dots,\omega^I\Big).$$

Theorem 6.3 (Local uniqueness; Du (Du n.d.), Thm. 3.3.2)**.** *The set* $\{\omega\in\mathbb{R}^{IL}_{++}\mid E(\omega)$ *is locally unique*$\}$ *is dense in* $\mathbb{R}^{IL}_{++}$.

Proof. We spell out the dimension count, the role of regular values, and the identification between the preimage of F and the equilibrium set.

Step 1: Dimensions of the domain and codomain of F*.* The domain of F is $S^{L-1}_{++}\times\mathbb{R}_{++}\times\mathbb{R}^{(I-1)L}_{++}$. Its dimension is

$$(L-1)+1+(I-1)L=IL,$$

which equals the dimension of the codomain $\mathbb{R}^{IL}$. Hence, if $\omega\in\mathbb{R}^{IL}_{++}$ is a regular value of F, the preimage theorem (Theorem 2.12) implies that $F^{-1}(\omega)$ is a smooth manifold of dimension

$$IL-IL=0,$$

provided it is non-empty. A zero-dimensional smooth manifold is locally modelled on open subsets of $\mathbb{R}^0$, i.e. on single points; it is therefore a discrete subset of the domain.

Step 2: Identification $F^{-1}(\omega)\leftrightarrow E(\omega)$*.* We show that discreteness of $F^{-1}(\omega)$ implies local uniqueness of equilibrium prices. Write $\omega=(\omega^1,\omega^2,\dots,\omega^I)$ and suppose

$$(p,w^1,\omega^2,\dots,\omega^I)\in F^{-1}(\omega).$$

By the definition of F, this means

$$f^1(p,w^1)+\sum_{i=2}^{I}f^i(p,p\cdot\omega^i)-\sum_{i=2}^{I}\omega^i=\omega^1. \tag{†}$$

Rearranging (†) gives

$$f^1(p,w^1)+\sum_{i=2}^{I}f^i(p,p\cdot\omega^i)=\sum_{i=1}^{I}\omega^i. \tag{‡}$$

Taking the inner product of both sides of (†) with p and using individual Walras's law gives

$$p \cdot \omega^1 = p \cdot f^1(p, w^1) + \sum_{i=2}^{I} p \cdot f^i(p, p \cdot \omega^i) - \sum_{i=2}^{I} p \cdot \omega^i = w^1 + \sum_{i=2}^{I} p \cdot \omega^i - \sum_{i=2}^{I} p \cdot \omega^i = w^1.$$

Thus $w^1 = p \cdot \omega^1$. Substituting this into (‡),

$$\sum_{i=1}^{I} f^i(p, p \cdot \omega^i) = \sum_{i=1}^{I} \omega^i,$$

which is precisely the Walrasian equilibrium condition $Z(p, \omega) = 0$. Hence $p \in E(\omega)$.

Conversely, if $p \in E(\omega)$, set $w^1 = p \cdot \omega^1$. The same algebra in reverse shows that $F(p, w^1, \omega^2, \ldots, \omega^I) = \omega$. Therefore equilibrium prices correspond exactly to points of $F^{-1}(\omega)$, with the auxiliary coordinate w^1 forced to equal $p \cdot \omega^1$.

Step 3: Density by Sard's theorem. Sard's theorem (Theorem 3.2) says that the set of critical values of F has Lebesgue measure zero in $\mathbb{R}^{IL}$. Consequently the set of regular values is dense in $\mathbb{R}^{IL}$: if an open ball consisted only of critical values, the set of critical values would have positive Lebesgue measure, contradicting Sard's theorem. Intersecting this dense set with the open set $\mathbb{R}^{IL}_{++}$ gives a dense subset of $\mathbb{R}^{IL}_{++}$. For every ω in this dense subset, $F^{-1}(\omega)$ is either empty or a zero-dimensional manifold, so by Step 2 the associated equilibrium set $E(\omega)$ is discrete — exactly the assertion of local uniqueness. □

Remark 6.4. *By additional arguments from differential topology, the number of equilibria is always odd (Dierker 1972).*

Remark 6.5 (Debreu–Mantel–Sonnenschein theorem and structural limits)**.** *A fundamental constraint on the apparatus of Sections 4–6 is imposed by the Debreu–Mantel–Sonnenschein (DMS) theorem (Sonnenschein (Sonnenschein 1972); Mantel (Mantel 1974); Debreu (Debreu 1974)): for any continuous function $Z\colon \Delta^{L-1}_{++} \to \mathbb{R}^L$ that is homogeneous of degree zero and satisfies Walras's law ($p \cdot Z(p) = 0$), there exists an exchange economy whose aggregate excess demand coincides with Z on a compact subset of Δ^{L-1}_{++}.*

This result has a direct topological consequence: the space of admissible aggregate excess demand functions is, modulo Walras's law and homogeneity, essentially unrestricted. In particular, the map $p \mapsto Z(p, \omega)$ may have any number of zeros and any configuration of their indices, so neither the number nor the stability pattern of equilibria can be determined from the aggregate data alone. The DMS theorem thus delineates the boundary of structural identification for the models of Sections 4–6: the topological tools (Sard's theorem, preimage theorem) establish generic local uniqueness but cannot overcome the non-identifiability of the global structure of Z.

7 Smale's Contribution

The Brouwer proof of existence is short and the Sard proof of local uniqueness is geometric, but neither tells the algorithm how to find the equilibrium. Smale's reformulation, also from the early 1970s, recasts the problem as a root-finding question on a one-dimensional manifold, where the solution becomes a path one can follow numerically. This is also the form in which existence and uniqueness can be treated jointly, by a single regularity argument on the same manifold.

Smale (Smale 1974, 1975) showed that the problem reduces to finding zeros of

$$z\colon \Delta_l \to \mathbb{R}^l,$$

using one-dimensional manifolds and Sard's theorem, without any convexity assumption. Here $\Delta_l = \{\mathbf{p} \in \mathbb{R}^{l+1}_+ : \sum_{h=0}^{l} p_h = 1\}$ is the standard price simplex (normalisation $\sum p_h = 1$), replacing the Euclidean price sphere S^{L-1}_+ used in Sections 4–6 (normalisation $\|p\| = 1$). The two spaces are diffeomorphic by $p \mapsto p/\|p\|$, so all equilibrium results carry over; Smale's formulation adopts the affine normalisation that is more convenient for the Global Newton Method.

Smale defines

$$\varphi^*(\mathbf{p}) = \varphi(\mathbf{p})/\|\varphi(\mathbf{p})\|, \quad \varphi(\mathbf{p}) = z(\mathbf{p}) - \mathbf{p}\left[\sum_h z^h(\mathbf{p})\right].$$

A connected component of the manifold $(\varphi^*)^{-1}(y)$, starting from the boundary $\partial\Delta_l$, must terminate in Σ — establishing existence of equilibrium. The Global Newton Method (Smale 1976) yields an algorithm for computing equilibrium starting from the boundary of the price simplex.

8 Social Choice and the Topology of Preferences

The Walrasian existence proof of Section 5 can be read as a topological statement: on a contractible space (the price simplex), every continuous self-map has a fixed point. Arrow's impossibility theorem for social choice has the opposite flavour: aggregating preferences continuously, anonymously, and in a way respecting unanimity is in general impossible. Chichilnisky's contribution was to recognise that the two results share a topological backbone: existence and impossibility both turn on the homotopy of the underlying preference space. The shift in this section is from differential to algebraic topology — homotopy groups, fixed-point indices — but the economic stakes are the same.

Chichilnisky (Chichilnisky 1993) reveals a deep connection: the topological obstruction to social choice and to the existence of equilibrium is one and the same obstruction.

Let X be a compact convex subset of $\mathbb{R}^l$, and P the space of smooth preferences on X. A social choice rule is a map $\phi: P^m \to P$. Chichilnisky and Heal (Chichilnisky and Heal 1983) proved the following:

Theorem 8.1 (Chichilnisky–Heal (Chichilnisky and Heal 1983)). *Let X be a connected CW complex (i.e. $\pi_0(X) = 0$). A continuous anonymous map $\phi: P^m \to P$ respecting unanimity exists for every $m \geq 1$ if and only if $\pi_j(X) = 0$ for all $j \geq 1$.*

Remark 8.2. (i) The role of connectedness. *The condition $\pi_j(X) = 0$ for all $j \geq 1$ is equivalent to contractibility of X only when X is also connected, i.e. $\pi_0(X) = 0$. Without connectedness the equivalence fails: a disjoint union of two contractible spaces satisfies $\pi_j = 0$ for every $j \geq 1$ yet is not contractible. Theorem 8.1 therefore requires the explicit hypothesis $\pi_0(X) = 0$, added above. The equivalence between vanishing of all homotopy groups and contractibility follows from Whitehead's theorem (see Hatcher (Hatcher 2002), Theorem 4.5): a map between CW complexes that induces isomorphisms on all homotopy groups is a homotopy equivalence; applying this to the map from X to a point gives contractibility once $\pi_0(X) = \pi_j(X) = 0$ for all $j \geq 1$.*

(ii) CW structure of the preference space. *The application of Whitehead's theorem requires X — or, in the context of Theorem 8.4 below, the preference space P — to carry a CW complex structure (or at least to be homotopy equivalent to a CW complex), and requires that the isomorphisms on homotopy groups be induced by a specific map (not merely an abstract group isomorphism): Whitehead's theorem says that a map $f: A \to B$ between CW complexes inducing isomorphisms $f_*: \pi_n(A) \xrightarrow{\sim} \pi_n(B)$ for all $n \geq 0$ is a homotopy equivalence. It does not assert that any abstract isomorphism $\pi_n(A) \cong \pi_n(B)$ implies $A \simeq B$.*

In the present setting X is a compact convex subset of $\mathbb{R}^l$, hence contractible and, being an ANR, homotopy equivalent to a CW complex (Milnor (Milnor 1959)). The canonical map from X to a point then induces isomorphisms on all homotopy groups (trivially, since X is already contractible), so the Whitehead conclusion is available. The space P of smooth preferences on X likewise inherits a suitable homotopy type under standard regularity assumptions (see Chichilnisky (Chichilnisky 1980)), but a full verification that a specific map $P \to \{\mathrm{pt}\}$ induces isomorphisms on all $\pi_n(P)$ requires the additional step of confirming that P is path-connected and simply connected (or more generally aspherical), which is not automatic from the definition and depends on the topology chosen on the space of preference relations. This step is stated here as a condition on P rather than a derived fact, in the interest of precision.

Chichilnisky (Chichilnisky 1995) defines the *preferred cone* of agent i with endowment $\omega_i \in \mathbb{R}^l$ as the set of directions of unbounded improvement:

$$A_i = \{v \in \mathbb{R}^l : \forall M > 0\ \exists \lambda > M : (\omega_i + \lambda v) \succ_i \omega_i\},$$

and its dual $D_i = \{w \in \mathbb{R}^l : w \cdot v \geq 0 \text{ for all } v \in A_i\}$. Condition (A): $\bigcap_{i=1}^m D_i \neq \emptyset$; Condition (B): $\bigcup_{i \in \theta} D_i$ is contractible for every coalition $\theta \subseteq \{1, \ldots, m\}$.

Remark 8.3. *The formula written in some survey treatments as* $A_i = \{v \in \mathbb{R}^l : \forall x \in \mathbb{R}^l\ \exists \lambda > 0 : (e_i + \lambda v) \succ_i x_i\}$ *is non-standard and erroneous on two counts. First,* e_i *there is used as a stand-in for the endowment* ω_i*, conflating the notation for a basis vector with that of an endowment point. Second, the quantification* $\forall x \in \mathbb{R}^l$ *is too strong: it would require agent i to prefer* $\omega_i + \lambda v$ *over every bundle in* $\mathbb{R}^l$*, an unsatisfiable condition for any* $\lambda > 0$ *whenever the preference order is locally non-satiated. The correct reading, following Chichilnisky (Chichilnisky 1995) Definition 1, captures instead the half-lines along which utility is* unbounded*:* $v \in A_i$ *if and only if moving arbitrarily far in direction v from* ω_i *eventually surpasses any fixed reference bundle, i.e.* $(\omega_i + \lambda v) \succ_i \omega_i$ *for all sufficiently large* λ*.*

Theorem 8.4 (Chichilnisky; Du (Du n.d.) via (Chichilnisky 1993))**.**

(i) *Conditions* (A) *and* (B) *are equivalent.*

(ii) *Condition* (A) *is necessary and sufficient for the existence of a competitive equilibrium.*

(iii) *Economy E has an equilibrium if and only if the preference space admits a continuous anonymous social choice rule respecting unanimity.*

The contractibility condition in Theorem 8.4 breaks down under increasing returns, where preferred cones need not be convex. Brown and Heal (Brown and Heal 1979) show that equity and efficiency can nevertheless be reconciled in economies with increasing returns provided the production set satisfies a suitable non-convex boundary condition, illustrating how topological obstructions to equilibrium interact with the geometry of production.

9 Topological Foundations of Location Theory

The two preceding sections worked with manifolds (differential topology) and homotopy (algebraic topology). Location theory needs neither: it needs a sensible notion of distance between parcels of land, where land is naturally modelled not as a finite-dimensional vector but as an element of a σ-algebra of measurable subsets. The Hausdorff metric provides that distance. With the Hausdorff metric the commodity space becomes topological in a non-trivial way, and otherwise sensible-looking demand correspondences acquire continuity properties that the finite-dimensional intuition does not predict.

Berliant and ten Raa (Berliant & ten Raa 1988) (with corrigendum (Berliant and ten Raa 1992)) propose a model with commodity space $\mathscr{B}$ = the σ-algebra of measurable subsets of a compact $L \subset \mathbb{R}^n$ (land). *Outer Hausdorff topology*: a basis of the topology on $\mathscr{B}$ is formed by sets of the form

$$\left\{C \in \mathscr{B} \;\middle|\; H\big((\mathring{C})^c, (\mathring{B})^c\big) < \delta,\ \left|\textstyle\int_C h\,dm - \int_B h\,dm\right| < \delta\right\},$$

where H is the Hausdorff metric on closed sets.

Theorem 9.1 (Berliant–ten Raa (Berliant & ten Raa 1988), Thm. 1)**.** *If* $U : \mathscr{B} \to \mathbb{R}$ *is continuous with respect to the above topology, then the consumer problem* $\max_{B\in\mathscr{B}} U(B)$ *subject to* $\int_B p\,dm \le y$ *has a solution.*

Theorem 9.2 (Berliant–ten Raa (Berliant & ten Raa 1988), Thm. 2)**.** *A continuous preference relation on equivalence classes* $\tilde{\mathscr{B}}$ *admits a continuous utility representation.*

10 Optimal Transport and Economics: Selected Results

Where location theory needed a topology on subsets, matching markets need a topology on probability measures. The bridge is optimal transport: the cost of moving one distribution to another defines a metric (the Wasserstein metric), and the corresponding minimisation problem turns out to be the correct framework for hedonic prices, marriage markets, and revealed-preference identification in matching. The Chiappori–McCann–Nesheim equivalence below is the cleanest statement of why optimal transport belongs here at all: three apparently distinct economic problems become one optimisation problem on the product type space.

10.1 Hedonic price equilibria, stable matching, and optimal transport

Chiappori, McCann and Nesheim (Chiappori and McCann and Nesheim 2010) establish the equivalence of three structures: (i) hedonic price equilibria under quasi-linear preferences, (ii) stable matchings with transferable utilities, (iii) the linear programme of optimal transport (Monge–Kantorovich). Under the twist condition the optimal matching is unique and pure. The twist condition itself states that the map $y \mapsto \nabla_x \Phi(x,y)$ is injective for every x: it ensures that the Φ-subdifferential of any dual potential is single-valued, which is the analytical counterpart of pure matching (Galichon 2016).

10.2 Falsifiability of incompletely specified models

Ekeland, Galichon and Henry (Ekeland and Galichon and Henry 2010) show that falsifiability of a model is equivalent to the existence of a zero-one-cost transport plan. The dual transport problem provides testable statistics for model specification testing; the general framework connecting partial identification with the Monge–Kantorovich duality is developed systematically in (Galichon 2016), Section 9.1.

10.3 The squared Bessel process and the ECIR model

Maghsoodi (Maghsoodi 1996) extends the CIR model to ECIR, allowing time-dependent parameters. The central object is the connection between ECIR and the squared Bessel process $\mathrm{BESQ}(\delta)$. The Bessel representation:

$$r(s) = \mathscr{K}^{-2}(t,s)\,\tilde{r}\Big(\tfrac{1}{4}\int_t^s \sigma^2(u)\mathscr{K}^2(t,u)\,du\Big),$$

where $\mathscr{K}(t,s) = \exp\frac{1}{2}\int_t^s k(u)\,du$ is a smooth, strictly positive time-change factor that defines a smooth monotone reparametrisation of the time axis.

Remark 10.1. *The function $\mathscr{K}(t,\cdot)\colon [t,\infty) \to \mathbb{R}_{>0}$ is smooth and strictly increasing (for locally integrable k), so the map $s \mapsto \frac{1}{4}\int_t^s \sigma^2(u)\mathscr{K}^2(t,u)\,du$ is a smooth, strictly monotone time change of the real line. This is sometimes loosely called a "diffeomorphism of the time axis," but that usage is imprecise in the stochastic-process context: a time change in the sense of Dambis–Dubins–Schwarz is an isomorphism of filtered probability spaces, not a diffeomorphism between smooth manifolds. The correct description is* smooth monotone reparametrisation of the time axis *(or* smooth time substitution*), which reduces the ECIR SDE to a* $\mathrm{BESQ}(\delta)$ *equation by a purely analytical change of variable, with no manifold structure invoked.*

Closed-form formula for a European bond option (Thm. 3.1 of (Maghsoodi 1996)):

$$\mathscr{C}(t) = P(t,T)\,\chi^2(\rho_2;\,\delta,\gamma_2) - K\,P(t,s)\,\chi^2(\rho_1;\,\delta,\gamma_1).$$

10.4 The implied volatility surface as a random field on a manifold

Cont and da Fonseca (Cont and da Fonseca 2002) model log-changes of the volatility surface as a stationary random field on A. Karhunen–Loève expansion:

$$X_t = \sum_n x_n(t)\,f_n.$$

On S&P 500 and FTSE data the first three modes explain over 95% of daily variance. The expansion defines a *local chart* around the mean I_0:

$$(x_1,\ldots,x_d) \mapsto I_0 \exp\Big(\sum_k x_k f_k\Big)$$

is a local diffeomorphism from $\mathbb{R}^d$ onto an open neighbourhood of I_0 in $\mathscr{M} \subset L^2(A)$.

10.5 Nonparametric IV regression as estimation on an infinite-dimensional manifold

Chen and Christensen (Chen and Christensen 2018) consider the NPIV model:

$$Y_i = h_0(X_i) + u_i, \quad E[u_i \mid W_i] = 0.$$

An important precursor for nonparametric demand analysis is Hausman and Newey (Hausman and Newey 1995), who develop nonparametric estimates of exact consumer surplus and deadweight loss, demonstrating that consistent nonparametric identification of welfare objects is feasible when demand is estimated without functional-form restrictions. The NPIV framework of Chen–Christensen extends this programme to the endogenous-regressor case. Optimal sup-norm convergence rates:

$$\|\hat{h} - h_0\|_\infty = O_p\Big((n/\log n)^{-(p-|\alpha|)/(2(p+\varsigma)+d)}\Big).$$

The sieve Ψ_J provides a sequence of finite-dimensional approximating subspaces whose increasing union is dense in $B_\infty(p,L)$; by analogy with a local atlas, each sieve space acts as a finite-dimensional coordinate patch. This analogy is heuristic: the Hölder ball $B_\infty(p,L) = \{h : \|h\|_{C^p} \leq L\}$ is a convex, bounded subset of the Banach space C^p, not a smooth manifold in the differential- topological sense, and the sieve approximation scheme does not by itself constitute an atlas in the sense of Definition 2.4.

Remark 10.2. *A rigorous manifold interpretation requires working in the framework of* Banach manifolds *(Abraham, Marsden and Ratiu (Abraham and Marsden and Ratiu 1988)): one must verify that the transition maps between sieve-coordinate patches are smooth in the Fréchet (or Gâteaux) sense and that the resulting atlas satisfies the compatibility conditions. For the Hölder ball this programme is non-trivial because $B_\infty(p,L)$ has a boundary (it is not an open subset of C^p), so it does not itself carry a manifold structure; one can at most treat the interior $\{h : \|h\|_{C^p} < L\}$ as an open subset of C^p, which is trivially a Banach manifold modelled on C^p. The sieve Ψ_J then provides finite-dimensional Galerkin approximations of the parameter space rather than genuine local charts. The differential-topological content of the Chen–Christensen result lies in the fact that convergence rates depend on the* intrinsic *(manifold) dimension of the parameter space, not on the ambient dimension of $\mathbb{R}^d$ in which data are observed — a genuinely geometric insight that does not require the full Banach-manifold apparatus to state.*

11 Differential Topology, Crises, and Contemporary Economics

The Sard-and-preimage machinery so far has been deployed to settle classical questions about Walrasian equilibrium. A more ambitious idea, present already in Smale's late writings and in Chichilnisky's, is to put a manifold structure on the space of economic *states* — the joint configuration of prices, production decisions, network exposures — and to read crises as critical points of a function on that manifold. This is where Morse theory enters: the change of Morse index at a critical point is the topological signature of a crisis. The section is admittedly programmatic — a fully specified Lyapunov–Morse function for a real economy is still an open problem — but the conceptual machinery is in place and the empirical analogues, persistent homology and contagion topology, are operational.

11.1 The economy as a smooth manifold

Definition 11.1 (Manifold of economic states)**.** *Let $\mathscr{S}$ be the set of all realisable states of the economy. We say $\mathscr{S}$ is a* smooth n-dimensional manifold *if every neighbourhood of every point $s \in \mathscr{S}$ is diffeomorphic to an open subset of $\mathbb{R}^n$.*

11.2 Morse theory and critical thresholds

Definition 11.2 (Economic Morse function)**.** *Let $V : \mathscr{S} \to \mathbb{R}$ be a smooth value function (e.g. GDP per capita or social welfare). V is a Morse function if all its critical points are non-degenerate.*

Proposition 11.3 (Topological interpretation of critical points; Milnor (Milnor 1965))**.** *Writing $M_c := V^{-1}((-\infty, c])$: if (a,b) contains no critical values then $M_b \cong M_a$; if (a,b) contains a single critical value of index k then $M_b \simeq M_a \cup e^k$.*

11.3 Persistent homology, curvature, and systemic instability

Definition 11.4 (Persistent homology of market data)**.** *For the sublevel filtration $\mathscr{S}_t = \{s \in \mathscr{S} \mid V(s) \leq t\}$,* persistent homology *tracks when homology classes are born and die as t varies.*

A second geometric indicator of fragility, Ollivier–Ricci curvature on networks, operates at the same scale. For an edge (i, j) in a weighted graph, the discrete Ricci curvature $\kappa(i, j)$ uses the L^1 Wasserstein distance between the probability measures μ_i and μ_j that spread unit mass from i and j over their respective neighbourhoods:

$$\kappa(i, j) = 1 - \frac{W_1(\mu_i, \mu_j)}{d(i, j)}.$$

Positive curvature on an edge indicates that the neighbourhoods of i and j are "close on average" relative to $d(i,j)$ — a sign of network redundancy and robustness to shocks. Sandhu, Georgiou and Tannenbaum (Sandhu and Georgiou and Tannenbaum 2016) apply this construction to equity correlation networks built from S&P 500 returns and show that the *Ricci flow* (the gradient flow $\partial_t w_{ij} = -\kappa(i,j)\, w_{ij}$) drives the network towards a uniform-curvature state. Empirically, the market-average Ollivier–Ricci curvature drops significantly in the months preceding the 2008 crash, providing a leading indicator of systemic fragility that is complementary to the persistent-H_1 norm of Gidea–Katz. The two indicators pick up complementary signals of the same underlying instability: curvature on the correlation network drops while new H_1 cycles emerge in the return cloud.

11.4 Connection to Walrasian equilibrium: a research programme

The standard static Walrasian model of Sections 4–6 assigns no dynamics to prices. To connect the Morse-theoretic apparatus of Subsection 11.2 to equilibria, one must introduce a price-adjustment process. The classical *tâtonnement* of Walras (see Balasko (Balasko 1988), Chapter 8) specifies the ordinary differential equation on $\mathscr{S} \subset S^{L-1}_{++}$:

$$\dot{p} = Z(p, \omega),$$

where the velocity of each price p_ℓ equals the aggregate excess demand $Z_\ell(p,\omega)$. Under this dynamics, the Walrasian equilibria $E(\omega)$ are precisely the *stationary points* of the flow, i.e. the zeros of the vector field $Z(\cdot,\omega)$.

If one further postulates a smooth function $V : \mathscr{S} \to \mathbb{R}$ that serves as a *Lyapunov function* for the tâtonnement — in the sense that $\dot{V}(p(t)) = \nabla V(p) \cdot Z(p,\omega) \leq 0$ along every trajectory, with equality only at equilibria — then the equilibria $E(\omega)$ become candidates for critical points of V. In that case the Morse-theoretic vocabulary of Proposition 11.3 is applicable: the index k of a critical point p^* encodes the dimension of the unstable manifold of the tâtonnement at p^*, so $k = 0$ corresponds to a (Morse) stable equilibrium and $k \geq 1$ to a saddle or unstable one.

Remark 11.5 (Status of the programme)**.** *The connection just described constitutes a research programme rather than a theorem, for three reasons.*

(i) Existence of a Lyapunov–Morse function. *No general construction of a smooth V satisfying both the Morse condition (all critical points non-degenerate) and the Lyapunov condition ($\dot{V} \leq 0$ along tâtonnement trajectories) is known for arbitrary exchange economies. Balasko (Balasko 1988) establishes stability results for the tâtonnement under gross substitutability, but even then the natural candidate $V(p) = \|Z(p,\omega)\|^2$ need not be a Morse function.*

(ii) Global vs. local. *Even when a local Lyapunov function exists near a regular equilibrium* $p^* \in E(\omega)$*, extending it to all of* $\mathscr{S}$ *consistently with the Morse conditions is non-trivial and may fail when multiple equilibria coexist.*

(iii) Static model. *Sections 4–6 work with a purely static model. Item (i) of the synthesis below invokes Sard's theorem for the map* $F : S^{L-1}_{++} \times \mathbb{R}^{IL}_{++} \to \mathbb{R}^{IL}$*, which is a statement about the parameter space* ω*, not about the dynamics on* $\mathscr{S}$*. Conflating the two requires the additional dynamical structure introduced above.*

The synthesis below is therefore stated as a programme: each item is either already proved (items marked with the relevant theorem) or describes the content of the proposed Morse–tâtonnement connection.

Proposition 11.6 (Morse–tâtonnement synthesis programme)**.** *Let* $\mathscr{S} \subset S^{L-1}_{++}$ *carry the tâtonnement* $\dot{p} = Z(p, \omega)$ *and suppose* $V : \mathscr{S} \to \mathbb{R}$ *is a smooth Lyapunov–Morse function for this flow. Then:*

(i) (Proved: Theorem 6.3) *The set of Walrasian equilibria* $E(\omega) \subset \mathscr{S}$ *is discrete for almost every* ω (Sard's theorem + preimage theorem).

(ii) (Programme) *Every* $p^* \in E(\omega)$ *is a stationary point of the tâtonnement and, under the Lyapunov–Morse hypothesis, a critical point of* V*.*

(iii) (Programme) *The Morse index* k *of each such* p^* *equals the dimension of its unstable manifold under the tâtonnement flow, classifying* p^* *as stable (*$k = 0$*), a saddle (*$0 < k < L-1$*), or unstable (*$k = L-1$*).*

(iv) (Programme) *Systemic crises correspond to transitions through saddle points of* V*, i.e. to the attachment of a cell* e^k *with* $k \geq 1$ *in the sublevel-set filtration of* V (Proposition 11.3).

12 Topological Data Analysis (TDA) in Contemporary Economics

The Morse-theoretic programme of the previous section is in part aspirational: a fully specified Lyapunov–Morse function on the state manifold of a real economy is not yet on the table. Topological data analysis turns the programme into an algorithm. Rather than guessing the Morse function and reading off its critical points, TDA takes the empirical point cloud — daily returns, daily network snapshots — and computes its topology directly: how many connected components, how many loops, how persistent each feature is across scales. The crisis-detection result of Gidea and Katz is the cleanest example of TDA earning its keep on real financial data.

Topological data analysis (TDA) is a field that systematically translates methods of algebraic topology — principally persistent homology — into algorithms operating on empirical data (Carlsson 2009). Since the mid-2000s, with the development of implementations (Ripser, Gudhi, Giotto-TDA), TDA has become operational. This section discusses four classes of TDA applications in economics, illustrating their connection to the apparatus of Sections 5–11.

12.1 Persistent homology of time series: detection of financial crises

Setup. Let $\{r_t\}_{t=1}^T$ be the vector of log-returns of d assets. For a rolling window $[t-w,t]$ we form the point cloud $P_t = \{r_s\}_{s=t-w}^t \subset \mathbb{R}^d$.

Definition 12.1 (Vietoris–Rips complex)**.** *For threshold $\varepsilon > 0$:*

$$\mathrm{VR}_\varepsilon(P_t) = \left\{\sigma \subset P_t : \max_{x,y\in\sigma} \|x-y\| \leq \varepsilon\right\}.$$

The family $\{\mathrm{VR}_\varepsilon(P_t)\}_{\varepsilon\geq 0}$ forms a filtration *of simplicial complexes.*

The homology groups $H_k(\mathrm{VR}_\varepsilon)$ change discontinuously as ε varies; the pair $(\varepsilon_{\text{birth}}, \varepsilon_{\text{death}})$ forms a *persistence point*; their collection is the *persistence diagram* $\mathrm{Dgm}_k(P_t)$.

Definition 12.2 (Persistence landscape; Bubenik (Bubenik 2015))**.** *From the persistence diagram $\mathrm{Dgm}_k(P_t)$ one constructs a countable family of functions $\lambda_s^{(k)} : \mathbb{R} \to \mathbb{R}$, $s = 1,2,\ldots$, called the* persistence landscape *of H_k. Here the superscript k indexes the homology group H_k and the subscript s orders the landscape functions by decreasing prominence ($s = 1$ being the dominant one). Following Gidea–Katz (Gidea and Katz 2018), the time-series indicator is the L^2 norm of the first landscape function ($s = 1$) of H_k computed in the rolling window ending at time t:*

$$L_k(t) = \|\lambda_{1,t}^{(k)}\|_{L^2(\mathbb{R})},$$

where the double subscript $(1,t)$ makes explicit that $s = 1$ (first landscape) and t (rolling window) are independent indices. This scalar serves as a one-dimensional topological indicator of the time series at time t.

Theorem 12.3 (Gidea–Katz (Gidea and Katz 2018), sketch)**.** *On S&P 500 data (2007–2009) and Dow Jones data (2015–2016), the statistic $L_1(t) = \|\lambda_{1,t}^{(1)}\|_{L^2(\mathbb{R})}$ (the first landscape function of H_1, i.e. the dominant cycle indicator) increases significantly 3–8 weeks before crashes, while the VIX shows no signal of comparable lead time.*

Read through Proposition 11.6, the growth of L_1 corresponds to the formation of persistent loops in the return cloud — formalising the build-up of circular correlations between assets and the approach to a saddle point of V.

12.2 Financial networks and topological phase transitions

Acemoglu, Ozdaglar and Tahbaz-Salehi (Acemoglu et al. 2015) analyse a financial system of n institutions connected by networks of mutual liabilities. Each institution can fail if its losses exceed its capital buffer; the failure of a node propagates losses to creditors, potentially triggering a cascade of bankruptcies. The parameter $\theta > 0$ denotes the size of the idiosyncratic shock hitting a single node initially.

Theorem 12.4 (Acemoglu–Ozdaglar–Tahbaz-Salehi; sketch (Acemoglu et al. 2015))**.** *There exists a threshold $\bar{\theta}$ such that:*

(i) *for shocks $\theta < \bar{\theta}$, a denser network better absorbs losses (risk diversification);*

(ii) *for shocks $\theta > \bar{\theta}$, a denser network faster propagates contagion (systemic contagion).*

The transition at $\bar{\theta}$ is topological: the connected components of the network, after removing bankrupt nodes, change structure discontinuously — analogously to, though not precisely the same as, a change of Morse index in Proposition 11.3.

Remark 12.5. *The analogy with Morse theory deserves careful qualification. Proposition 11.3 concerns a smooth Morse function $V : \mathscr{S} \to \mathbb{R}$ on a smooth manifold: passing a critical value of index k attaches a cell e^k and changes the topology of the sublevel set $V^{-1}((-\infty, c])$. The phenomenon in Theorem 12.4 is categorically different in nature: it is a combinatorial property of a graph. When nodes are removed from the liability network (as institutions fail), the number of connected components of the graph can jump discontinuously; this is a feature captured by H_0 of the Vietoris–Rips complex or, directly, by graph-connectivity arguments, not by the Morse index of a smooth critical point. The structural parallel is that both phenomena involve a discontinuous topological transition in a parameter-indexed family — a change in π_0 of the network on the one hand, and the attachment of a handle of index k on the other — but the mathematical tools required are distinct: Morse theory for the smooth setting, persistent H_0 or graph theory for the combinatorial one. Calling the network transition "a change of Morse index" without this qualification would be an abuse of terminology.*

12.3 Labour markets and optimal matching

Galichon and Salanié (Galichon and Salanié 2022) develop a labour market model in which the worker–firm pair (x, y) generates surplus $\Phi(x, y) + \varepsilon(x, y)$, and the equilibrium is identical to an optimal transport plan:

$$\max_{\mu \in \mathscr{M}(\mathbf{P}, \mathbf{Q})} \int \Phi(x, y)\, d\mu(x, y).$$

The Monge–Kantorovich duality theorem guarantees that any solution chosen by a central planner can be decentralised as an equilibrium and vice versa, providing a welfare theorem for matching models with transferable utility (Galichon 2016). Key result: identification of Φ from matching data is possible if and only if the dual transport problem has a unique solution — which depends on the topology of the supports $\mathbf{P}$ and $\mathbf{Q}$ (cf. (Chiappori and McCann and Nesheim 2010)). This is a direct analogue of Theorem 8.4 transposed to an empirical context.

Remark 12.6. *The Galichon–Salanié model is the current standard in empirical research on occupational segregation and the marriage market. Implementation: packages `TraME` (R) and `PyTRAME` (Python).*

12.4 The manifold hypothesis and high-dimensional econometrics

The results of Chen–Christensen (Subsection 10.5) and Cont–da Fonseca (Subsection 10.4) are special cases of the manifold hypothesis: high-dimensional economic data concentrate on a smooth submanifold $\mathscr{M} \subset \mathbb{R}^d$ of dimension $m \ll d$. Nonparametric convergence rates depend on the intrinsic dimension m, not on d. The topological foundation of this observation — how many connected components, loops and higher holes $\mathscr{M}$ has — is now addressed by TDA (Carlsson 2009; Bubenik 2015).

13 Conclusions

Empirical application	**Topological tool**	**Source**
Crash detection (S&P 500)	Persistent homology H_1; landscapes	Gidea–Katz (2018)
Financial networks; contagion	Connected components of liability network	Acemoglu et al. (2015)
Labour market matching ID	OT; Monge–Kantorovich duality	Galichon–Salanié (2022)
NPIV estimation (sup norm)	Sieves on Hölder manifold	Chen–Christensen (2018)
Implied vol. surface dynamics	KL expansion on manifold $L^2(A)$	Cont–da Fonseca (2002)
Dimensionality reduction	Barcodes H_k; manifold hypothesis	Carlsson (2009)

Table 1: Contemporary applications of topological tools in empirical economics and finance. Compiled by the author.

Economic result	**Topological tool**	**Source**
Existence of equilibrium	Kakutani's fixed-point thm.	Arrow–Debreu (1954)
Existence of equilibrium	1-dim. manifolds + Sard	Smale (1974, IIA)
Local uniqueness	Sard + preimage thm.	Debreu (1970); Smale (1974, IIA)
Increasing returns (exist.)	Homotopy groups; cones	Chichilnisky (1990)
Computing equilibrium	Global Newton method	Smale (1976)
Social choice	Homotopy groups π_j	Chichilnisky (1980)
Equil. ⇔ soc. choice	Contractibility of cones D_i	Chichilnisky (1993)
Location theory	Outer Hausdorff metric	Berliant & ten Raa (1988)
Interest rate models (ECIR)	Process BESQ(δ); time change	Maghsoodi (1996)
Implied vol. surface dynamics	Karhunen–Loève expansion	Cont–da Fonseca (2002)
Nonparametric IV regression	Sieves on Hölder spaces	Chen–Christensen (2018)
Crash detection (empirical)	Persistence landscapes H_1	Gidea–Katz (2018)
Contagion in bank networks	Topology of liability graph	Acemoglu et al. (2015)
Labour market matching	OT; Monge–Kantorovich duality	Galichon–Salanié (2022)
TDA / barcodes	Persistent homology H_k	Carlsson (2009)

Table 2: Summary of main results and topological tools. Compiled by the author. Smale published a series of at least fifteen papers under the title *Global Analysis and Economics* (1973–1981); IIA refers to Part IIA (*J. Math. Econ.* 1(1), 1974, pp. 1–14), which proves existence without the Kakutani machinery and finiteness of equilibria from Sard. The original Arrow–Debreu proof uses the Kakutani fixed-point theorem (Kakutani 1941) because aggregate excess demand is a correspondence; an alternative Brouwer-based proof was given only by Geanakoplos (2003).

The classical applications listed in the upper part of Table 1 and the corresponding rows of the summary table — existence of competitive equilibrium, finiteness of regular equilibria, the homotopy obstruction to social choice, the structure of the equilibrium manifold — have no purely analytic substitutes. Chichilnisky's (Chichilnisky 1993) observation that the equilibrium existence problem and the social choice problem rest on the same obstruction (contractibility of preferred cones) was the strongest unification of two of these threads.

The contemporary half of the table looks different. It is no longer about establishing that a fixed point exists; it is about reading topological invariants off data. Persistence landscape norms of H_1 precede the 2008 and 2015–2016 crashes by three to eight weeks (Gidea and Katz 2018). The cascading-failure threshold in the Acemoglu–Ozdaglar–Tahbaz-Salehi model is governed by which components of the liability graph survive bankruptcy of a single node (Acemoglu et al. 2015). Identification in matching reduces to uniqueness of a transport plan over a specific topological support (Galichon and Salanié 2022). Sup-norm rates in nonparametric IV depend on the intrinsic dimension of the parameter manifold rather than the ambient dimension (Chen and Christensen 2018).

The two halves of the table are not in competition. The classical results gave the language — regular values, homotopy classes, contractibility, Hausdorff distance, the manifold hypothesis — in which the contemporary ones can even be stated. What has changed is the cost of computing the relevant invariants: software libraries (Ripser, POT, TraME) have made operations that were once theoretical accessible on standard datasets.

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