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**Abstract**

In this paper an alternative approach to modelling and forecasting single asset returns volatility is presented. A new, bivariate, flexible framework, which may be considered as a development of single-equation ARCH-type models, is proposed. This approach focuses on joint distribution of returns and observed volatility, measured by Garman-Klass variance estimator, and it enables to examine simultaneous dependencies between them. Proposed models are compared with benchmark GARCH and range-based GARCH (RGARCH) models in terms of prediction accuracy. All models are estimated with maximum likelihood method, using time series of EUR/PLN spot rate quotations and WIG20 index. Results are very encouraging especially for foreasting Value-at-Risk. Bivariate models achieved lesser rates of VaR exception, as well as lower coverage tests statistics, without being more conservative than its single-equation counterparts, as their forecasts errors measures are rather similar.

**Keywords:**
bivariate volatility models, joint distribution, range-based volatility estimators, Garman-Klass estimator, observed volatility, volatility modelling, GARCH, leverage, Value-at-Risk, volatility forecasting

**JEL:**
C13, C32, C53, C58, G10, G17

## 1. Introduction

Volatility modeling is at the forefront of financial econometrics interest. Still growing importance of this subject comes from both business and regulatory institutions of financial markets. Over past three decades, dozens of models have been proposed. All of them address to a very specific challenges of volatility modeling like leptokurtosis of empirical returns distribution, volatility clustering or asymmetry effect. There is a common belief that volatility is predictable, at least to a certain degree. Models built to forecast volatility are called conditional volatility models, because they try to infer about future volatility conditional on present set of informations. Generally, they can be divided into two competing groups: Autoregressive Conditional Heteroskedasticity (ARCH) models pioneered by Engle (1982) and Stochastic Volatility (SV) models. This paper focuses on the former group. Its aim is to propose a flexible framework that not only enhances forecasting performance of ARCH-type models, but also allows to draw some conclusions about relationship between asset returns and its observed volatility, measured by some kind of variance estimator. The main feature of proposed approach is focusing on joint distribution of returns and observed volatility by using dual-equation structure. Such an approach is possible due to use of more efficient range-based daily variance estimators instead of squared returns (or innovations) as a volatility proxy.

In this paper two financial time series are investigated: EUR/PLN spot rate quotation and WIG20 index. Four conditional volatility models are employed to obtain volatility of assets returns predictions. These are well known GARCH model and its range-based counterpart (RGARCH), as well as two newly developed bivariate models derived from GARCH and RGARCH models respectively. Bivariate models show very promising performance especially in terms of forecasting Value-at-Risk. Moreover, they allow to examine simultaneous dependencies between observed volatility and returns.

The rest of paper is organized as follows. Section 2 briefly describes volatility models that are most relevant from this paper's point of view, as well as reviews volatility estimators based on high, low, open and close prices (range-based estimators). Section 3 contains derivation of proposed models. In section 4 empirical results are presented for both in-sample and out-of-sample analysis. Section 5 concludes.

## 2. Literature review

The main difference between Stochastic Volatility and Autoregressive Conditional Heteroskedasticity models is an assumption about the nature of volatility: in case of ARCH-type models volatility is considered as a deterministic process, whereas in case of SV models volatility has a fully stochastic nature. Regardless of chosen approach, researchers try to incorporate an asymmetry effect, when negative shocks in asset returns have different impact on future volatility than positive ones. In case of SV models an asymmetry effect is usually examined by introducing the correlation between innovations in returns and volatility. This kind of asymmetry is often called a leverage effect and was described by Harvey and Shephard (1996). It should be noticed that leverage effect excludes cases when negative and positive innovations have an impact with the same sign and differ only in magnitude. Asai and McAleer (2005) developed a more general asymmetric SV model that does not impose such a restriction and can accomodate both leverage and size effect. Despite its conceptual attractiveness, Stochastic Volatility models are not as popular as its ARCH-type counterparts. The main reason for this is the fact that SV models are, in general, computationally demanding, as their likelihood can not be obtained in closed form. There are many methods of parameters estimation in Stochastic Volatility models, however not all of them are feasible in case of models with leverage effect. Harvey and Shephard proposed Quasi-Maximum

Likelihood Estimation (QMLE) method and employed Kalman filter to obtain quasi-likelihood function. The comprehensive description of SV models estimation techniques can be found in Broto and Ruiz (2004).

Autoregressive Conditional Heteroskedasticity models are widely used among practitioners, mostly due to their flexibility and straightforward estimation. Since seminal Engle's paper, dozens of ARCH-type models have been proposed, usually they differ only in parameterization of conditional variance equation. Arguably, the most important derivation of ARCH model is a Generalized Autoregressive Conditional Heteroskedasticity model proposed by Bollerslev (1986) which forms a basis for almost all modern ARCH-type models, mostly due to its flexible framework and relatively low number of parameters.

There are several ways to incorporate an asymmetry effect in ARCH-type models, but two of them are especially popular. The first one is to consider separately positive and negative squared innovations: this approach has been proposed by Glosten, Jaganathan and Runkle (1993) and formally established as a GJR-GARCH model. The second approach is to formulate conditional variance equation in exponential form and allow for arbitrary dependency between volatility and lagged, standardized innovations. Exponential form of equation guarantees that even if negative correlation between volatility and returns occurs, conditional variance is still greater than zero. Such a model, allowing to capture various kind of asymmetry, including leverage effect, was derived by Nelson (1991) and is widely known as an EGARCH (Exponential GARCH) model. Both aforementioned models focus on how past returns influence present volatility. However, an implication in opposite direction may also occur: increased level of (conditional) volatility can raise probability of negative returns. Such an effect is usually examined by using GARCH-in-Mean models (GARCH-M), where present conditional variance is used as a regressor in conditional mean equation.

In classical framework, ARCH-type models demand only time series of asset close prices. Recently, models using additional variables are becoming more popular. They base on assumption that there exist better volatility proxies than simple squared close-to-close returns (innovations). Indeed, using high, low, close and open (HLCO) daily prices, one can obtain more efficient variance estimators than squared daily returns. Those estimators are often called range-based, most important of them were developed by Parkinson (1980), Garman and Klass (1986) and Rogers and Satchell (1992), and they are given by following formulas:

$$\sigma^2_{Park,t} = \frac{1}{4\ln 2}(\ln H_t - \ln L_t)^2 \qquad (1)$$

$$\sigma^2_{GK,t} = 0.5(\ln(H_t / L_t))^2 - (2\ln 2 - 1)(\ln(C_t / O_t))^2 \qquad (2)$$

$$\sigma^2_{RS,t} = \frac{1}{N}\sum_{n=t-N}^{t}\ln(H_n / O_n)(\ln(H_n / O_n) - \ln(C_n / O_n)) + \ln(L_n / O_n)(\ln(L_n / O_n) - \ln(C_n / O_n)) \qquad (3)$$

In above formulas *H, L, C* and *O* are respectively: highest, lowest, close and open price.

The common feature of range-based daily estimators is that they are up to 10 times more efficient than simple squared daily return. However, in empirical works those estimators turn to be downward biased due to discrete nature of observed asset prices.

It was a matter of time before range-based estimators have been used in volatility modelling. This pioneering research was conducted by Alizadeh, Brandt and Diebold (2001). In their paper range-based Stochastic Volatility model was proposed. Authors found results encouraging mostly due to useful distributional property of range (logarithm of range is approximately Gaussian) that improves performance of QMLE method. A different approach was chosen by Chou (2005). He examined dynamic behaviour of range and formulated a Conditional Autoregressive Range (CARR) model. CARR model is a member of Multiplicative Error Models class. Using CARR model conditional volatility is obatined in two steps: first conditional range is predicted, then forecast of volatility is computed by inserting conditional range into Parkinson formula (1). The main weakness of CARR model is that it focuses only on range, ignoring asset returns distribution, thus it often underestimate returns variance. The most widespread range-based volatility model is certainly REGARCH (Range-based Exponential GARCH) proposed by Brandt and Jones (2006). Authors used aforementioned distributional property of log-range and reformulated conditional variance equation of EGARCH model in following way:

$$R_t = 0.43 + \ln h_t + 0.29\varepsilon_t$$

$$h_t = \exp(\varpi + \alpha(\frac{R_{t-1} - 0.43 - \ln h_{t-1}}{0.29}) + \beta \ln h_{t-1} + \delta_1 \frac{r_{t-1}}{h_{t-1}} \quad (4)$$

$$\varepsilon_t \sim N(0,1)$$

Where $R$ is a logarithm of range, $r$ is daily return and $h$ is conditional variance.

A different approach was chosen by Lildholdt (2003). Author leaves conditional variance equation unchanged in comparison to classic GARCH(1,1) model, but estimate model parameters using joint distribution of vector of maximal, minimal and close (HLC) prices. The exact formula for density function of HLC prices distribution is complicated and would not be presented in this paper. Moreover it contains infinite sum, thus require some truncation and may be difficult in implementation. Recently an extension of model proposed by Lildhold have been developed by Fiszeder and Perczak (2013). Authors not only use joint distribution of HLC prices, but also modify conditional variance equation inserting custom range-based variance estimator in place of squared innovations.

Over the last few years, several others range-based ARCH-type models have been developed (e.g. Molnar (2011), Skoczylas (2013, 2014)). Most of them show rather promising performance when compared to their return-based counterparts.

## 3. Models derivation

Certain parametrisations of single equation ARCH-type models are able to investigate lagged dependencies like: assymetry, leverage effect, or influence of conditional volatility on present returns. However, what single equation ARCH-type models can not do, is to capture simultaneous dependency between returns and observed volatility. A natural way to incorporate such an effect is to treat observed volatility (measured by variance estimator) as a random variable and focus on joint distribution of returns and observed volatility.

In classical ARCH framework, returns are assumed to be normally distributed with conditional mean $\mu_t$ and conditional variance $h_t$. Conditional mean is usually modeled as an ARMA process, however due to simplicity a constant mean is assumed in this paper:

$$r_t = \mu + \varepsilon_t$$
$$\varepsilon_t \sim N(0, h_t) \quad (5)$$

It is necessary to make some assumptions on observed volatility distribution. The first and the most intuitive one is that observed volatility is a noisy approximation of returns volatility. In this paper, Garman-Klass estimator is used as an observed volatility measure. The second assumption is that observed volatility distribution is approximately log-normal. Empirical results shows that range-based variance estimators are indeed well described by log-normal distribution (e.g. Alizadeh et al. (2001)). Moreover, in most option pricing models, from which proposed approach draws some inspirations, volatility is assumed to be distributed log-normally. Under these assumptions, relationship between observed and conditional volatility may be expressed in following way:

$$\begin{aligned}&\sigma^2_{GK,t} = Kh_t\xi_t \\ &\xi_t \sim \ln N(0, v_t)\end{aligned} \qquad (6)$$

Where $\sigma^2{}_{GK}$ is an observed volatility measured by Garman-Klass estimator and $h$ is conditional variance of returns. A constant $K$ is included to capture potential bias in Garman-Klass estimator. Taking logarithms of both sides leads to:

$$\ln \sigma^2_{GK,t} = k + \ln h_t + \eta_t \qquad (7)$$

Where $\eta$ has Gaussian distribution with zero mean. The next step is to investigate joint distribution of $\varepsilon$ and $\eta$. They are both normally distributed with zero mean, thus their joint distribution is fully described by their covariance matrix:

$$\begin{bmatrix}\varepsilon_t \\ \eta_t\end{bmatrix} \sim N\left(\begin{bmatrix}0 \\ 0\end{bmatrix}, \begin{bmatrix} h_t & \rho\sqrt{h_t v_t} \\ \rho\sqrt{h_t v_t} & v_t\end{bmatrix}\right) \qquad (8)$$

Now it is quite easy to see some similarities between proposed framework and Heston option pricing model (1993). A correlation $\rho$ between $\varepsilon$ and $\eta$ is set to be constant, while variance of $\eta$ is assumed to be time-varying and follow ARCH(1) process:

$$v_t = v_0 + v_1\eta^2_{t-1} \qquad (9)$$

To obtain full parameterization of model, it is necessary to plug conditional variance equation. The main advantage of proposed approach is that any conditional variance equation can be chosen. In this paper, conditional variance equation from much celebrated GARCH(1,1) model is used (10), therefore, the proposed model will be called BGARCH (Bivariate GARCH):

$$h_t = \varpi + \alpha\varepsilon^2_{t-1} + \beta h_{t-1} \qquad (10)$$

Additionally, alternative form of conditional variance equation is analyzed. This equation comes from RGARCH(1,1) model and may be expressed as follow:

$$h_t = \varpi + \alpha\sigma^2_{GK,t-1} + \beta h_{t-1} \qquad (11)$$

Analogically, this model will be called BRGARCH.

It is worth to notice, that equation (11) already uses range-based variance estimator (Garman-Klass) as a volatility proxy, so it is a very interesting question whether BRGARCH model will show any significant improvement over RGARCH. There exists one theoretical advantage of BRGARCH model over RGARCH. In BRGARCH model, it is possible to obtain formula for unconditional variance. One should notice that in single equation RGARCH model unconditional variance can not be determined, because in general:

$\mathrm{E}\sigma^2{}_{GK,t} \neq h_t$.

However, in BRGARCH model, using properties of log-normal distribution, equation (6) and (9), it could be shown that:

$$E\sigma^2_{GK,t} = EKh_t\xi_t = Kh_tE\xi_t = K\exp(Ev_t/2)h_t = \exp(k + 0.5v_0/(1-v_1))h_t \qquad (12)$$

Thus, unconditional variance in BRGARCH model may be expressed as:

$$Var(\varepsilon_t) = \frac{\varpi}{1 - \exp(k + 0.5v_0/(1-v_1))\alpha - \beta} \qquad (13)$$

Now it follows, that sufficient condition for covariance stationarity of $\varepsilon$ is:

$$\exp(k + 0.5v_0/(1-v_1))\alpha + \beta < 1 \qquad (14)$$

It should be underlined that such an inference was possible due to bivariate nature of BRGARCH model, and could not be conducted in a single equation RGARCH model.

Using well known properties of bivariate normal distribution it is possible to determine conditional distribution of $\eta$ given $\varepsilon$:

$$\eta_t \mid \varepsilon_t \sim N(\rho \varepsilon_t \sqrt{\frac{v_t}{h_t}}, (1-\rho^2)v_t) \qquad (15)$$

A sign of conditional expectation of $\eta_t$ given $\varepsilon_t$ depends solely on signs of $\varepsilon_t$ and $\rho$. Knowing $\rho$, one can find how present returns affect present observed volatility. In equities and securities markets increased volatility usually occurs during periods of downward trend, thus one should expect negative values of $\rho$. It is less so clear in case of foreign exchange markets, where convention of quotation plays crucial role. Generally, if base currency is considered to be stronger than counter currency, the pair rates tend to follow upward trend during turbulent periods – in such a case one should expect positive values of $\rho$. The opposite conclusion holds when a reverse relation between currencies occurs.

## 4. Data and results

Daily data including open, high, low and close prices are used. The data set is obtained from financial website stooq.pl and it covers period from 1.01.2008 to 31.12.2014. Two time series are examined: EUR/PLN spot rate and Warsaw Stock Exchange WIG20 index. Logarithmic returns are analysed.

In the first step, in-sample analysis is conducted. Models were estimated for the whole analysed period (from 1.01.2008 to 31.12.2014). Maximum likelihood estimates of parameters of four aforementioned models are presented for EUR/PLN (table 1) and WIG20 (table 2) respectively.

**Table 1.** Parameters estimates for EUR/PLN spot rate.

| | GARCH | BGARCH | RGARCH | BRGARCH |
|---|---|---|---|---|
| $\mu$ | -0.0095 | 0.0039 | -0.0019 | -0.0024 |
| | 0.353 | 0.725 | 0.856 | 0.828 |
| $\varpi$ | **0.0030** | **0.0064** | -0.0021 | **0.0045** |
| | 0.003 | 0.000 | 0.270 | 0.000 |
| $\alpha$ | **0.1099** | **0.1213** | **0.2808** | **0.2742** |
| | 0.000 | 0.000 | 0.000 | 0.000 |
| $\beta$ | **0.8882** | **0.8681** | **0.7681** | **0.7474** |
| | 0.000 | 0.000 | 0.000 | 0.000 |
| $\rho$ | - | **0.1402** | - | **0.1127** |
| | | 0.000 | | 0.000 |
| $v_0$ | - | **0.5056** | - | **0.4678** |
| | | 0.000 | | 0.000 |
| $v_1$ | - | 0.0324 | - | 0.0393 |
| | | 0.129 | | 0.078 |
| $k$ | - | **-0.4001** | - | **-0.3568** |
| | | 0.000 | | 0.000 |

Looking at these tables, some patterns become evident. Coefficient $\alpha$ is slightly larger, whereas $\beta$ slightly smaller in BGARCH, comparing to GARCH, which indicates that BGARCH model is more responsive to recent innovations. In both cases parameter $\varpi$ is insignificant in RGARCH model, while the same parameter in BRGARCH is significant and comparable in magnitude to GARCH and BGARCH ones. What is encouraging, is that parametres $\rho$, $v_0$, $v_1$ and $k$ estimates are quite similar, regardless of which conditional variance equation is used. According to expectation, parameter $\rho$ is negative for stock exchange index, and positive for EUR/PLN pair (as it is obvious that EUR is considered as stronger currency than PLN). For both assets, parameter $v_1$ estimates are small (in case of EUR/PLN even insignificant at 0.05 confidence level), thus variance of $\eta$ seems to be rather constant over time, at least for analysed time series.

**Table 2.** Parameters estimates for WIG20.

| | GARCH | BGARCH | RGARCH | BRGARCH |
|---|---|---|---|---|
| $\mu$ | 0.0053 | -0.0214 | -0.0326 | -0.0261 |
| | 0.848 | 0.451 | 0.247 | 0.357 |
| $\varpi$ | **0.0156** | **0.0303** | 0.0092 | **0.0290** |
| | 0.009 | 0.000 | 0.350 | 0.000 |
| $\alpha$ | **0.0698** | **0.0888** | **0.1872** | **0.2692** |
| | 0.000 | 0.000 | 0.000 | 0.000 |
| $\beta$ | **0.9242** | **0.8973** | **0.8872** | **0.8294** |
| | 0.000 | 0.000 | 0.000 | 0.000 |
| $\rho$ | - | **-0.1811** | - | **-0.1777** |
| | | 0.000 | | 0.000 |
| $v_0$ | - | **0.5139** | - | **0.4943** |
| | | 0.000 | | 0.000 |
| $v_1$ | - | **0.0120** | - | **0.0095** |
| | | 0.008 | | 0.013 |
| $k$ | - | **-0.8101** | - | **-0.8078** |
| | | 0.000 | | 0.000 |

Using well known formulas for returns-based GARCH models, as well as recently derived equation (13), unconditional variances may be computed:

**Table 3.** Unconditional variances.

| | GARCH | BGARCH | RGARCH | BRGARCH |
|---|---|---|---|---|
| EUR/PLN | 1.560 | 0.599 | - | 0.568 |
| WIG20 | 2.579 | 2.172 | - | 1.750 |

As mentioned before, in case of RGARCH models it is not possible to obtain unconditional variance. It is worth to notice that unconditional variance calculated using GARCH parameters estimates is highest in both cases.

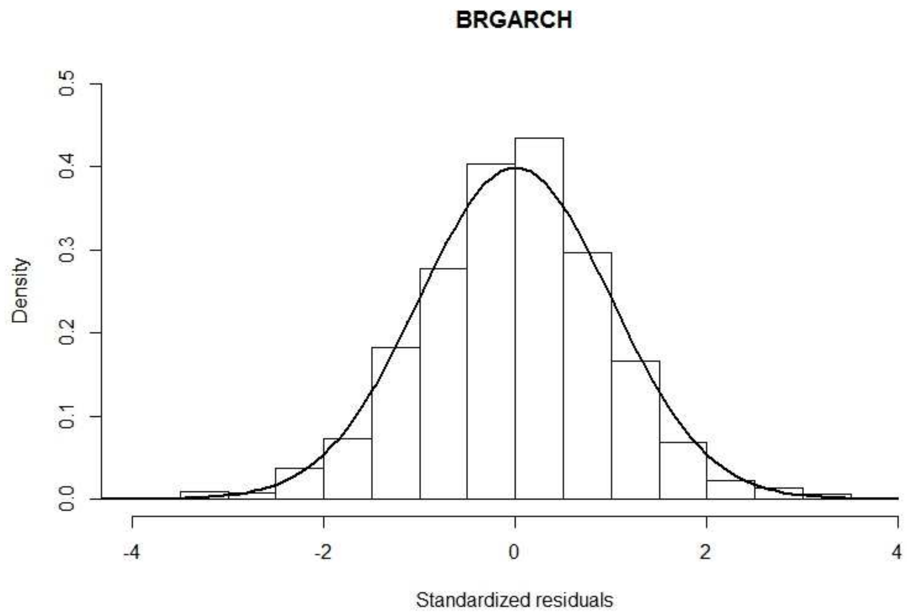
Though several diagnostic tests for ARCH-type models can be conducted, two of them are mainly popular. These are tests for autocorrelation of squared, standardized residuals, and normality of standardized residuals. Their results are presented in table 4. In all except one cases (RGARCH for WIG20), models seem to deal with volatility clustering phenomena, as p-values of Ljung-Box test are greater than 0.05. All models fail to pass test for normality of standardized residuals.

**Table 4.** Ljung-Box and Jarque-Bera tests results.

| | GARCH | BGARCH | RGARCH | BRGARCH |
|---|---|---|---|---|
| EUR/PLN | | | | |
| Ljung-Box test | 3.386 | 2.928 | 3.625 | 4.853 |
| p-value | 0.641 | 0.711 | 0.605 | 0.434 |
| Jarque-Bera test | 52.09 | 55.86 | 20.24 | 23.05 |
| p-value | 0.000 | 0.000 | 0.000 | 0.000 |
| WIG20 | | | | |
| Ljung-Box test | 8.245 | 8.157 | 14.972 | 10.183 |
| p-value | 0.143 | 0.148 | 0.010 | 0.070 |
| Jarque-Bera test | 97.75 | 94.65 | 112.42 | 107.66 |
| p-value | 0.000 | 0.000 | 0.000 | 0.000 |

In figure 1, histogram of standardized residuals from BRGARCH model estimated for WIG20 is presented, along with standard normal density curve. Visual assesment suggests that more fat-tailed distribution could be more appropriate. However, one should keep in mind that in case of BGARCH and BRGARCH standardized residuals distribution is in fact marginal distribution, as both bivariate models assume joint normality of $\varepsilon$ and $\eta$.

**Figure 1.** Histogram of standardized residuals from BRGARH model estimated for WIG20.




In the second step of research, out-of-sample analysis is conducted. All four models were estimated on a rolling window of 750 observations. Each time, every model was estimated using most recent 750 observation and one-day-ahead volatility forecasts were obtained. Analysed period again cover 1.01.2008 to 31.12.2014, thus data set was enlarged to include 750 necessary observations prior to 1.01.2008.

A standard way to assess forecasting performance is to calculate forecast errors measures, which is not an issue provided that true values of forecasted variable are available. It is well known that exact volatility can not be observed, thus measuring volatility forecast errors heavily rely on volatility proxy. Patton (2010) thoroughly reviews several forecast errors measures and finds that particularly two loss function seem to be more robust to noise in volatility approximations than others. These are: Mean Squared Error, and QLIKE function given by following formula:

$$L(\sigma^2, h) = \ln(h) + \sigma^2 / h \qquad (16)$$

Where $\sigma^2$ is observed volatility measured by some kind of variance estimator, and $h$ is a volatility forecast.

It should be noticed that QLIKE is an asymmetric loss function, thus it tends to favor models that overestimate rather than underestimate true volatility. In this paper, two kind of observed daily volatility measures are used: squared daily return and Garman-Klass estimator.

Both MSE and QLIKE measures were computed for one-day-ahead volatility forecasts obtained from all four analyzed models. In table 5 values of aforementioned loss functions are presented, values computed using Garman-Klass estimator as volatility proxy are marked with asterisk (*).

**Table 5.** Values of loss functions, out-of-sample analysis.

| | *GARCH* | *BGARCH* | *RGARCH* | *BRGARCH* |
|---|---|---|---|---|
| *EUR/PLN* | | | | |
| *MSE* | *1.1583* | *1.1635* | *1.1541* | *1.1571* |
| *MSE(*)* | *0.4204* | *0.4201* | *0.3821* | *0.3792* |
| *QLIKE* | *-0.2262* | *-0.2239* | *-0.2569* | *-0.2551* |
| *QLIKE(*)* | *-0.3059* | *-0.3050* | *-0.3278* | *-0.3281* |
| *WIG20* | | | | |
| *MSE* | *30.9895* | *31.2269* | *31.4655* | *32.0069* |
| *MSE(*)* | *8.7258* | *8.5677* | *9.8766* | *10.0666* |
| *QLIKE* | *1.6123* | *1.6136* | *1.6031* | *1.5992* |
| *QLIKE(*)* | *1.2099* | *1.2070* | *1.1937* | *1.1857* |

Range-based models seem to outperform their return-based counterparts in case of EUR/PLN spot rate. The picture is less clear for WIG20 index where GARCH and BGARCH models are better in terms of MSE. Pairwise comparison between GARCH and BGARCH, as well as RGARCH and BRGARCH, shows that their values of loss functions are pretty similar.

Many researchers argue that while forecasting volatility, particular emphasis should be placed on ability to properly predict tail observations, thus an ultimate test of model's forecasting performance should be computing Value-at-Risk and backtesting. In this paper VaR at *99%* level was computed using one-day-ahead volatility predictions from analyzed models. After counting VaR exceptions, tests for coverage accuracy were conducted: one for unconditional coverage (Kupiec test) and second for conditional coverage (Christoffersen test). The null hypothesis of Kupiec test is that observed VaR exceptions at *(1-p)%* level, over the period of $n$ days, come from binomial distribution with parameters $p$ and $n$. This test focuses on frequency of VaR exceptions. Christoffersen test additionally tests whether VaR exceptions are independent. Full results of backtesting procedure are presented in table 6. Proposed models perform much better than their single-equation counterparts. For both assets, using bivariate versions of models leads to lower VaR exception rate, as well as decreases values of tests statistics. It is worth to notice, that in case of EUR/PLN spot rate, transition from single- to dual equation model is sufficient to pass both coverage tests.

**Table 6.** Results of backtesting $VaR_{0.99}$ computed using out-of-sample volatility forecasts.

| | GARCH | BGARCH | RGARCH | BRGARCH |
|---|---|---|---|---|
| EUR/PLN | | | | |
| % of $VaR_{0.99}$ breaches | 1.616% | 1.393% | 1.616% | 1.337% |
| unconditional coverage test | 5.792 | 2.492 | 5.792 | 1.863 |
| p-value | 0.0161 | 0.1144 | 0.0161 | 0.1723 |
| conditional coverage test | 7.199 | 3.940 | 7.199 | 3.334 |
| p-value | 0.0273 | 0.1395 | 0.0273 | 0.1888 |
| WIG20 | | | | |
| % of $VaR_{0.99}$ breaches | 1.770% | 1.599% | 1.542% | 1.485% |
| unconditional coverage test | 8.541 | 5.372 | 4.458 | 3.619 |
| p-value | 0.0035 | 0.0205 | 0.0347 | 0.0571 |
| conditional coverage test | 19.061 | 12.774 | 22.004 | 16.411 |
| p-value | 0.0000 | 0.0017 | 0.0000 | 0.0003 |

Another interesting question is a behavior of parameter $\rho$ across out-of-sample period. In figures 2 and 3 point estimates of $\rho$ for both bivariate models are presented. It becomes apparent that, at least in case of EUR/PLN spot rate, parameter $\rho$ significantly fluctuate over time, thus assumption of constant correlation between $\varepsilon$ and $\eta$ should be repealed. Allowing for time-varying correlation will be a subject of later research, as it requires reformulation of both BGARCH and BRGARCH models. Analyzing graphs 2 and 3 one should remember that these are not fitted values of correlation at time $t$, but a parameter values obtained from estimating model on 750 observations prior to $t$.

**Figure 2.** Parameter $\rho$ estimates for EUR/PLN spot rate across time.

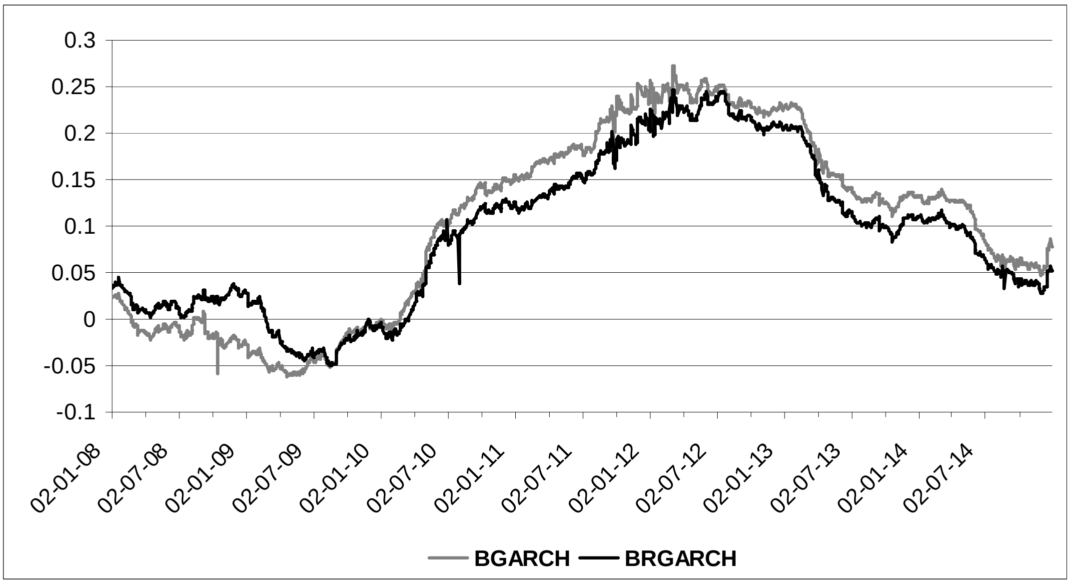
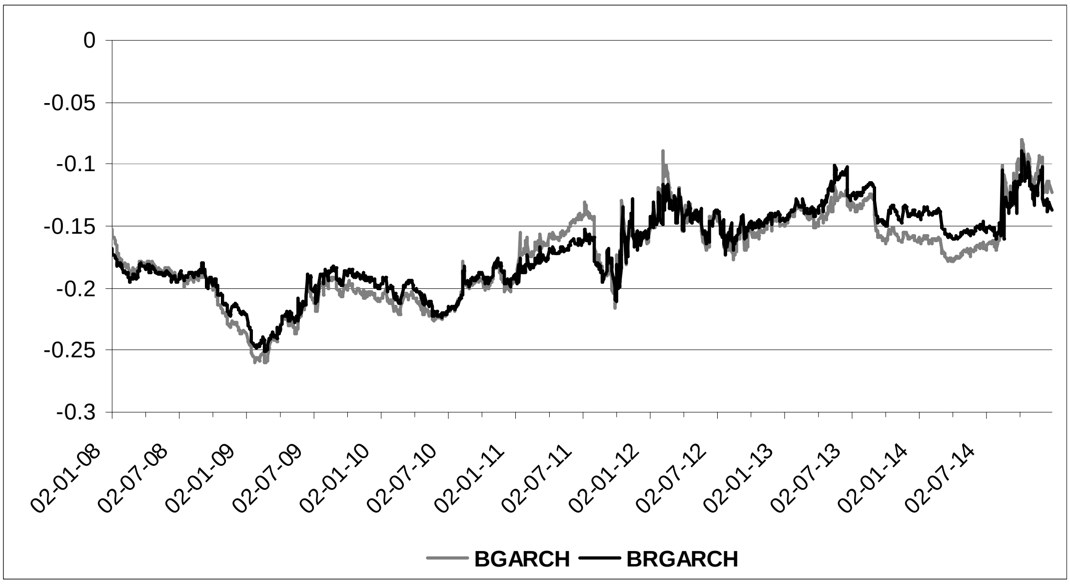


**Figure 3.** Parameter $\rho$ estimates for WIG20 index across time.




## 5. Conclusions

In this paper a new approach to modelling volatility is proposed. The main feature of this approach is a dual equation structure that allows to model joint distribution of returns and their observed volatility. The first equation is an ARCH-type conditional variance equation, while the second one describes the relationship between observed and conditional volatilities. Proposed framework is very flexible, as it can be modified to incorporate any ARCH-type conditional variance equation. In this paper, equations coming from GARCH and RGARCH (Range-based GARCH) models are used. Efficient, range-based, Garman-Klass variance estimator is used as an observed volatility approximation. All models are estimated using time series of EUR/PLN spot rate and WIG20 index. Pairwise comparison between single- and dual equation models is conducted. Bivariate models do not differ significantly from their single-equation counterparts in terms of forecasting errors measures, however, they much better cope with Value-at-Risk forecasting, resulting in lesser rate of VaR exceptions, as well as lower values of coverage tests statistics.

Approach presented in this paper enables to find a solution of certain theoretical problem associated with Range-based GARCH models. In single equation RGARCH model, it is not possible to obtain unconditional variance, because the expected value of volatility proxy used is unknown. Incorporating RGARCH conditional variance equation into proposed bivariate framework solves this problem, as it allows to determine unconditional variance and infer about stationarity of asset returns process.

Due to assumed joint normality of returns and logarithms of observed volatility, it is possible to investigate simultaneous dependency between returns and observed volatility. The findings are in line with expectation: in equities markets a correlation between returns and observed volatility is negative, while in case of foreign exchange markets the sign of correlation depends on relative strength of currencies. Empirical results from out-of-sample analysis indicate that aforementioned correlation tends to fluctuate, thus a further development of proposed models is required to incorporate time-varying correlation.

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