



---

UNIVERSITY OF WARSAW

**Faculty of Economic Sciences**

---

**WORKING PAPERS**

No. 14/2013 (99)

**PAWEŁ WNUK LIPINSKI**

**PORTFOLIO SELECTION MODELS BASED  
ON CHARACTERISTICS OF RETURN  
DISTRIBUTIONS**

WARSAW 2013



## **Portfolio selection models based on characteristics of return distributions**

**PAWEŁ WNUK LIPINSKI**  
Faculty of Economic Sciences,  
University of Warsaw  
e-mail: wnuklipinski.pawel@gmail.com

### **Abstract**

This article concerns the problem of optimal portfolio selection. The objective of this paper is to indicate the best method and criteria for optimal portfolio selection. In order to achieve the objective six models including such optimization criteria as mean, variance, skewness, kurtosis and transaction costs are analyzed. The method of fuzzy multi-objective programming is used to transform multiple conflicting criteria into a single objective problem and to find optimal portfolios. In order to indicate the best portfolio selection model a simulation based on five years data from January 1, 2007 to December 31, 2011 was conducted. The portfolios were constructed from WIG20 stocks and WIBID 3M as risk-free asset.

### **Keywords:**

optimal portfolio, portfolio selection, fuzzy multi-objective programming, skewness, kurtosis

### **JEL:**

G11, C61

### **Acknowledgments:**

The original version of the paper was prepared as a master thesis. The author would like to thank his supervisors, Dr Paweł Sakowski and Dr Robert Ślepaczuk, for their valuable support and assistance during the research. Insightful comments provided by the participants of the seminar “Modelling and Forecasting Returns and Volatility on Capital Markets” are also gratefully acknowledged.

## 1. Introduction

The appropriate choice of securities to the portfolio is an important issue for the asset management on financial markets. Since the foundation of the Warsaw Stock Exchange in 1991 we can observe a dynamic development of the Polish financial market<sup>1</sup>. During the last ten years savings of Polish citizens more than doubled<sup>2</sup> and the capitalization of the WSE increased fourfold<sup>3</sup>. Thus, the society is becoming richer and people face the problem of choosing the best way to invest their money. The choice of securities to the portfolio can be done with the application of portfolio selection models. The use of these models can help to answer the question – in which and how many securities of each kind to invest. In this paper several portfolio selection models are analysed, which include such criteria as average rate of return, the risk quantified as variance and kurtosis, transaction costs and skewness, which is a measure of distribution asymmetry.

In this study the following two research hypotheses are set up:

- Portfolio optimization models allow obtaining better than average results.
- Addition of higher moments to the standard mean-variance model improves the results.

Moreover, attempts are made to answer the following research questions:

- Does the addition of risk-free rate to portfolio of stocks improve performance of the models?
- Does constraining transaction costs in the optimization process translate into better results?

In order to verify the hypotheses and answer the research questions an empirical study was conducted. Six models including criteria based on portfolio distribution of returns are compared. The models are compared on the basis of a simulation that was performed for a five years period from January 1, 2007 to December 31, 2011. The portfolios are composed of stocks from WIG20 index and WIBID3M and are reconstructed each quarter. The method of fuzzy multi-objective programming is used to transform multiple conflicting criteria into a single-objective problem and to find the optimal portfolio.

The remainder of the paper is structured as follows. After introduction we come to the literature review in the second section. In the third section the methodology and data of the research are described, as well as the assumptions of the study are formulated. The fourth section is devoted to the main results of the simulation. In this section the problems of incorporating transaction costs as optimization constraint and of adding a risk-free asset to the portfolio are analysed. In the fifth section the results of a broad sensitivity analysis are presented. The influence of boundary constraints to a position in one security and of change in the moment and frequency of portfolio reconstruction on the results is analysed. Subsequently, the impact of transaction costs on models' performance is presented. The paper ends with the indication of the optimal models' characteristics and conclusions.

---

<sup>1</sup> [www.gpw.pl/historia](http://www.gpw.pl/historia), access July 20, 2012.

<sup>2</sup> [www.wyborcza.biz/finanse/1,105684,12113202,Oszczednosci\\_Polakow\\_rosna\\_\\_W\\_bankach\\_mamy\\_prawie.html](http://www.wyborcza.biz/finanse/1,105684,12113202,Oszczednosci_Polakow_rosna__W_bankach_mamy_prawie.html), access July 20, 2012.

<sup>3</sup> [www.gpw.pl/analizy\\_i\\_statystyki](http://www.gpw.pl/analizy_i_statystyki), access July 20, 2012.

## 2. Literature review

The question of how to optimally select assets to a portfolio has its long history. It started in 1952 when Harry Markowitz published his ground-breaking paper entitled “Portfolio Selection”. This paper was a milestone in the development of modern portfolio theory and its practical applications. Markowitz assumes that the investor considers expected return as a desirable thing and risk as an undesirable thing. Return is quantified as the mean and risk as the variance of the rates of returns of securities. The investors are assumed to look for a balance between the maximization of the rate of return and minimization of risk of their investment decisions (Markowitz, 1952). The introduction of mathematical representation of risk and return made possible the application of optimization tools in the portfolio management problems. Since the investors dislike risk, they will choose a portfolio that maximizes return, given a fixed level of risk. Otherwise, the investors will choose a portfolio with a minimum risk, given a fixed level of return. This relationship is called the mean-variance efficiency and a portfolio that fulfils this condition is called an efficient portfolio. In other words, if a portfolio is inefficient then there exists another portfolio with either smaller variance and no smaller mean, or with larger mean and no larger variance. The set of all efficient portfolios forms the efficient frontier. Since the investors may have different preferences toward required risk and return, their optimal portfolios might be different, but their choices must be on the efficient frontier (Wang and Xia, 2002).

Since the seminal paper of Markowitz (1952), in which the mean-variance model was proposed, numerous studies concerning portfolio theory were published. The Markowitz model gained widespread acceptance as a tool for portfolio selection and revolutionized the way people think about the choice of assets to portfolio. However, there is still a debate among researchers whether higher moments than variance should be included in the process of portfolio selection. Many researchers (e.g. Arditti, 1971; Samuelson, 1970; Konno and Suzuki, 1995; Jean, 1971; Scott and Horvath, 1980) claim that portfolio analysis should be extended to higher moments unless the investors’ utility function is quadratic and the assets returns are normally distributed. Moreover, Scott and Horvath (1980) show that for investor that is consistent in the direction of preference of moments, the preference direction is positive for positive values of each odd central moment and negative for each even central moment. It means that investors should prefer high skewness and low kurtosis.

Contrary to the assumptions of the mean-variance model, there is ample evidence (Arditti, 1971; Tang and Shum, 2003; Chunchinda et al., 1997; Prakash et al., 2003) showing that securities and portfolio returns are not normally distributed. The asymmetry of the distribution is measured by the third moment – skewness. When skewness is positive it indicates that the right tail of the distribution is long in comparison to the left tail. For the investment portfolio it translates into high but rare gains and small but more frequent losses, assuming that mean is near zero. Generally, investors would prefer a portfolio with higher third moment, when the first and the second moments are the same. Furthermore, they can even prefer a portfolio with a larger skewness at the expense of larger variance and smaller mean (Konno et al., 1993). Research papers concerning the application of higher moments in portfolio selection (e. g. Konno and Suzuki, 1995; Lai, 1991; Chunchinda, 1997; Ryoo, 2007; Prakash et al., 2003) concentrate mainly on the first three moments i.e. the mean-variance-skewness framework. Empirical studies suggest that the incorporation of skewness into an investor’s portfolio might result in an improved optimal portfolio (Ryoo, 2007; Joro and Na, 2006). The fourth moment, kurtosis, is often omitted by researchers, though it is also important for portfolio selection, especially when the return distribution is non-normal. Kurtosis reflects the probability of extreme events. The

higher is the kurtosis, the larger is the probability of extreme events. Therefore it seems to be reasonable to take into account kurtosis in the portfolio selection process, which was applied in e. g. Lai et al. (2006) or Yu and Lee (2011).

As investors might consider different factors than mean and variance, according to their preferences, the choice of a flexible and comprehensive portfolio selection method is very important. Polynomial goal programming (Lai et al., 2006; Chunnachinda et al., 1997; Lai, 1991; Prakash et al., 2003) and fuzzy multi-objective programming (Zimmerman, 1978; Lee and Li, 1993; Yu and Lee, 2011) are methods that allow selecting optimal portfolio in a multi-objective framework. These methods allow the decision maker to choose portfolio selection criteria and then to find optimal portfolio that best suits their preferences. In this paper the application of the fuzzy multi-objective programming in portfolio selection is analysed.

Although many studies were conducted in the field of modern portfolio theory, the question which model and which criteria should be included in the portfolio selection is still open to debate. The appearance of more and more powerful computers and more efficient algorithms for solving complex optimization problems suggests that portfolio theory will develop in the direction of more advanced multi-objective models. This paper aims to add value to this strand of literature and analyse the rationale of applying multi-objective models in portfolio selection.

### 3. Methodology and data

#### 3.1 Data description

Data used in the study cover the period from June 30, 2006 to February 14, 2012. All the analysed time series are daily closing prices. All the data were downloaded from the website of Polish data provider [www.stooq.pl](http://www.stooq.pl). Stocks of companies included in the WIG20 index are analysed in the research. For a given point of time only the stocks of companies that were at this time in WIG20 index could have been chosen. This means that when the WIG20 constituents changed also the stocks that were analysed changed. Therefore for a given point of time always 20 stocks were analysed. Such an approach allowed avoiding survivorship bias. Additionally, the stock prices were adjusted by splits and dividends to better reflect real investment value.

Table 1. Constituents of WIG20 index on June 30, 2006

PKN ORLEN	TP S.A.	PEKAO S.A.	KGHM	PKO BP
BANK BPH	AGORA	BZ WBK	PGNIG	PROKOM
NETIA	TVN	LOTOS	BRE	MOL
GTC	KĘTY	BORYSZEW	BIOTON	ORBIS

\* Table 1 presents constituents of WIG20 index on June 30, 2006. The information was obtained from the website of the Warsaw Stock Exchange, [www.gpw.pl](http://www.gpw.pl).

In Table 1 the constituents of WIG20 index at the beginning of the analysed period are presented, whereas Table 2 shows the composition of the WIG20 index at the last day of the analysed period. During the time of more than 5 years 50% of the index was subject to change, which is a significant value.

Table 2. Constituents of WIG20 index on February 14, 2012

PKN ORLEN	TP S.A.	PEKAO S.A.	KGHM	PKO BP
GETIN	PGE	BANK HANDLOWY	PGNIG	ASSECOPOL
PBG	TVN	LOTOS	BRE	BOGDANKA
GTC	PZU	KERNEL	TAURON	CEZ

\* Table 2 presents constituents of WIG20 index on February 14, 2012. The information was obtained from the website of the Warsaw Stock Exchange, [www.gpw.pl](http://www.gpw.pl).

Additionally, the data of WIG index, Allianz Akcji FIO mutual fund and Allianz OFE pension fund for the same period are used. They serve as benchmarks to the models' results. These particular funds are chosen, because they were among the best funds in the analysed period in their categories – Allianz Akcji FIO among mutual funds and Allianz OFE among pension funds. As an approximation of risk-free rate WIBID3M – 3 month Warsaw Interbank Bid Rate – is applied.

### 3.2 Model specification

In the research six multi-objective models, which include criteria of return, variance, skewness, kurtosis and transaction costs, are analysed. The aforementioned criteria are based mainly on the characteristics of the distributions of the portfolio components' returns. Apart from that transaction costs are subject to analysis since it is supposed that the incorporation of transaction costs in the optimization process might improve the results. The analysed multi-objective models are the M (mean), the MV (mean, variance), the MS (mean, skewness), the MVS (mean, variance, skewness), the MVK (mean, variance, kurtosis) and the MVSK (mean, variance, skewness, kurtosis). The mathematical representation of the models, which is presented in the remainder of this subsection, is similar to the one shown in paper by Yu and Lee (2011) with some necessary modifications.

The M model consists of two objectives, the maximization of portfolio return and the minimization of transaction costs, as it is shown by equations (3.1) and (3.2). The detailed representation of the M model is as follows:

$$\text{Max } \sum_{i=1}^n r_i w_i, \quad (3.1)$$

$$\text{Min } \sum_{i=1}^n p(l_i + s_i), \quad (3.2)$$

$$\text{s.t. } \sum_{i=1}^n (w_i + p l_i + p s_i) = 1, \quad (3.3)$$

$$0 \leq w_i \leq 0,2, \quad (3.4)$$

for  $i=1, \dots, n$ ,

where  $n$  is the number of available securities;  $r_i$  is the return on security  $i$ ;  $w_i$  is the weight of the security  $i$  in the portfolio;  $p$  is the transaction cost expressed in percentages;  $l_i$  is the ratio of the value of  $i$  securities bought by the investor to the portfolio value;  $s_i$  is the ratio of the value of  $i$  securities sold by the investor to the portfolio value.

Constraint (3.2) represents the transaction costs that are paid by each portfolio reconstruction. The costs are a linear function of the ratio of bought and sold securities' value to

the portfolio value. Thus, the minimization of transaction costs is equivalent to minimization of portfolio turnover. Constraint (3.3) shows the available budget allocated to investment in securities and transaction costs of buying and selling. Obviously, the product of  $l_i$  and  $s_i$  ( $l_i s_i$ ) is equal to zero since a given security is either bought or sold. Constraint (3.4) represents the lower and the upper bounds of the total position in each security. This constraint indicates that short selling is not allowed in the model. Such an assumption is set up because on the Warsaw Stock Exchange short selling is allowed only from July 1, 2010<sup>4</sup>. Therefore the research would be unrealistic when in the analysed period the possibility of selling securities short would exist. Additionally the maximum share of a single security in the portfolio is assumed to be 20% so as the model does not concentrate too much on few securities. This assumption is a subject to change in the sensitivity analysis (subsection 5.1).

The MV is a triple objective model containing all the criteria of the M model and additionally an objective of variance minimization (3.5). The specification of the model is as follows:

$$\text{Max } \sum_{i=1}^n r_i w_i, \quad (3.1)$$

$$\text{Min } \sum_{i=1}^n p(l_i + s_i), \quad (3.2)$$

$$\text{Min } \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}, \quad (3.5)$$

s. t. Constraints (3.3) and (3.4),

where  $\sigma_{ij}$  is the covariance between security  $i$  and security  $j$ . When  $i=j$  then  $\sigma_{ij}$  represents the variance of the security.

The MS is a triple objective model including the criteria of the M model additionally with the objective of skewness maximization (3.6). The model is represented as follows:

$$\text{Max } \sum_{i=1}^n r_i w_i, \quad (3.1)$$

$$\text{Min } \sum_{i=1}^n p(l_i + s_i), \quad (3.2)$$

$$\text{Max } E \left( \frac{w^T (r - \bar{r})}{\sigma} \right)^3, \quad (3.6)$$

s. t. Constraints (3.3) and (3.4),

where  $E$  is the expected value operator; superscript  $T$  is the transposing operator;  $w = (w_1, \dots, w_n)$  is the vector of portfolio weights;  $r = (r_1, \dots, r_n)^T$  is the vector of securities returns;  $\bar{r} = (\bar{r}_1, \dots, \bar{r}_n)^T$  is the vector of expected returns;  $\sigma$  is the standard deviation of the portfolio returns.

The MVS is a model with four objectives, which includes all the criteria of MV model additionally with the objective of skewness maximization. The specification of the MVS is as follows:

$$\text{Max } \sum_{i=1}^n r_i w_i, \quad (3.1)$$

$$\text{Min } \sum_{i=1}^n p(l_i + s_i), \quad (3.2)$$

---

<sup>4</sup>[www.gpw.pl/o\\_spolce](http://www.gpw.pl/o_spolce), access May 26, 2012.

$$\text{Min } \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}, \quad (3.5)$$

$$\text{Max } E \left( \frac{w^T(r-\bar{r})}{\sigma} \right)^3, \quad (3.6)$$

s. t. Constraints (3.3) and (3.4),

The MVK is a model with four objectives, including all the criteria of MV model and additionally a criterion of kurtosis minimization (3.7). The MVK is represented as follows:

$$\text{Max } \sum_{i=1}^n r_i w_i, \quad (3.1)$$

$$\text{Min } \sum_{i=1}^n p(l_i + s_i), \quad (3.2)$$

$$\text{Min } \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}, \quad (3.5)$$

$$\text{Min } E \left( \frac{w^T(r-\bar{r})}{\sigma} \right)^4, \quad (3.7)$$

s. t. Constraints (3.3) and (3.4),

The MVSK is the most complex model with five objectives of return maximization (3.1), costs minimization (3.2), variance minimization (3.5), skewness maximization (3.6) and kurtosis minimization (3.7). The MVSK is represented as follows:

$$\text{Max } \sum_{i=1}^n r_i w_i, \quad (3.1)$$

$$\text{Min } \sum_{i=1}^n p(l_i + s_i), \quad (3.2)$$

$$\text{Min } \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}, \quad (3.5)$$

$$\text{Max } E \left( \frac{w^T(r-\bar{r})}{\sigma} \right)^3, \quad (3.6)$$

$$\text{Min } E \left( \frac{w^T(r-\bar{r})}{\sigma} \right)^4, \quad (3.7)$$

s. t. Constraints (3.3) and (3.4),

The above described multi-criteria models are solved by the method of fuzzy multi-objective programming. This approach presented in papers by Zimmermann (1978), Lee and Li (1993) and Yu and Lee (2011) allows to transfer a multi-objective model into a single-objective model. The fuzzy multi-objective programming uses the concept of fuzzy sets and the aspiration level  $\lambda$ . At first the program forces each objective to achieve its aspiration level, and then it provides a trade-off between conflicting criteria. The method requires ideal and anti-ideal values of variables to be provided in advance. The reformulation of a multi-objective model into a single-objective model by the use of fuzzy multi-objective programming is presented using the MV model as an example:

$$\text{Max } \lambda$$

$$\text{s. t. } \lambda \leq \frac{r^* - r_a}{r_i - r_a}, \quad (3.8)$$

$$\lambda \leq \frac{\sigma^* - \sigma_a}{\sigma_i - \sigma_a}, \quad (3.9)$$

$$\lambda \leq \frac{c^* - c_a}{c_i - c_a}, \quad (3.10)$$

$$r^* = \sum_{i=1}^n r_i w_i, \quad (3.11)$$

$$\sigma^* = \sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}}, \quad (3.12)$$

$$c^* = \sum_{i=1}^n p(l_i + s_i), \quad (3.13)$$

$$\sum_{i=1}^n (w_i + p l_i + p s_i) = 1, \quad (3.3)$$

$$0 \leq w_i \leq 0,2, \quad (3.4)$$

for  $i=1, \dots, n$ ,

where  $r^*$  is the return of the portfolio;  $r_a$  is the anti-ideal return of the portfolio;  $r_i$  is the ideal return of the portfolio;  $\sigma^*$  is the risk of the portfolio measured as standard deviation;  $\sigma_a$  is the anti-ideal risk of the portfolio;  $\sigma_i$  is the ideal risk of the portfolio;  $c^*$  is the transaction cost of the portfolio;  $c_a$  is the anti-ideal transaction cost;  $c_i$  is the ideal transaction cost.

Constraints (3.8)-(3.10) respectively present the goals of simultaneously maximizing return, minimizing risk and minimizing transaction costs of the portfolio. Here the ideal and anti-ideal values of the criteria must be provided. In this research it is assumed that the ideal and anti-ideal values for mean, variance, skewness and kurtosis of the portfolio are calculated as the best and worst historical observations taken from all the analysed securities at every rebalancing. The ideal value of transaction costs is 0 and the anti-ideal is 0,8%, which is 0,4% multiplied by 2, whereas 0,4% is the most common value of transaction costs charged by Polish brokerage houses in case of individual investors<sup>5</sup>. Thus in the research the 0,4% transaction cost was assumed as the most appropriate one. In this case the anti-ideal value of 0,8% means that at rebalancing the whole composition of the portfolio is changed. The other five models are transformed into single-objective models through fuzzy multi-objective programming in a similar way that was shown for the MV model.

### 3.3 Research assumptions

The main part of the research is a simulation of an investment which is based on the models presented in subsection 3.2. The investment period is assumed to be 5 years, from January 1, 2007 to December 31, 2011. The initial investment value is 1 million Polish Zloty and the portfolio can consist of 20 biggest Polish stocks and a risk-free rate. In the main part of the research the holding period of the portfolio is 3 months and the historical data used for estimation of models' parameters is 6 months. The maximum share of one company's stocks in the portfolio is set to 20%, but there is no such constraint for the risk-free rate, which means that the maximum share of risk-free rate in the portfolio is 100%. In the sensitivity analysis the maximum share is changed to 10% and 40%. Thus, on January 1, 2007 a portfolio of stocks is constructed based on parameters estimated on data from July 1, 2006 to December 31, 2006. Subsequently the

---

<sup>5</sup> The information is based on the summary of Polish brokerage houses' charges, found on the website <http://wyborcza.biz/Gieldy/55,114507,10068911,,,10069460.html>, access June 1, 2012.

portfolio is held 3 months without rebalancing. On the April 1, 2007 the portfolio is reconstructed based on parameters estimated on data from October 1, 2006 to March 31, 2007. Then this portfolio is held without changes till July 1, 2007. This process is continued till the end of 2011. Such a simulation is conducted for each of the six models. Additionally, different versions of the models are compared.

In the main research:

- Models including transaction costs in the optimization process are compared with models without this constraint;
- Models consisting only of WIG20 stocks are compared with models consisting of WIG20 stocks and risk-free rate.

Whereas in the sensitivity analysis:

- The maximum share of one company in the portfolio is changed from 20% to 10% and 40% for each model;
- The starting point of the rebalancing is moved from the beginning of the quarter to the middle of the quarter for all the models.
- The frequency of portfolio reconstruction is increased from 3 months to 1 month.

In order to better compare the models the following statistics were applied:

- Annual Return Compounded (ARC), calculated according to the following formula (Feibel, 2003):

$$ARC = 252 \frac{1}{N} \sum_{t=1}^N R_t$$

where: 252 is the assumed number of transaction days in a year;  $R_t$  is a daily logarithmic log-return.

- Annual Standard Deviation (ASD), expressed by the formula (Feibel, 2003):

$$ASD = \sqrt{252} \sqrt{\frac{1}{N} \sum_{t=1}^N (R_t - \bar{R})^2}$$

- Return to Risk ratio, a simple measure of return per unit of risk (Culp, 2001):

$$RR = \frac{ARC}{ASD}$$

- Average transaction cost, calculated in the following way:

$$average\ transaction\ cost = \frac{1}{NR} \sum_{t=1}^{NR} transaction\ cost_t$$

where: transaction cost<sub>t</sub> is transaction cost at time t expressed as % of invested capital, NR is the number of times the portfolio is reconstructed during the simulation period.

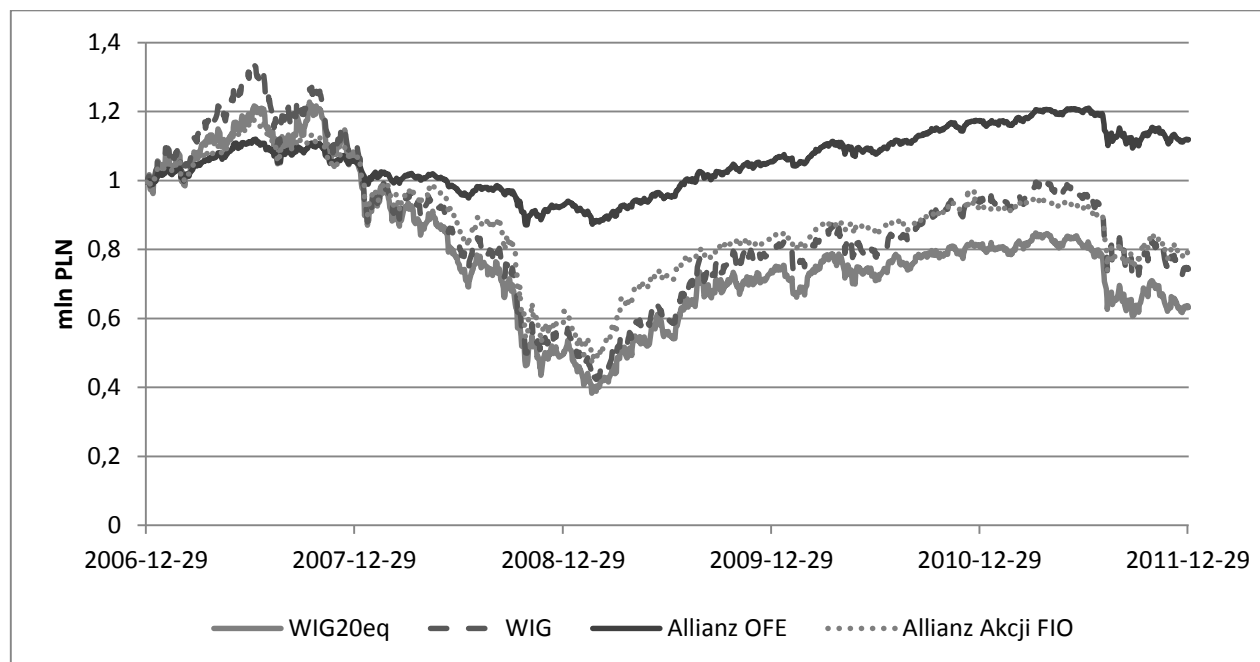
Transaction costs are included in all the models analysed in this study. The results obtained for each model are compared with the performance of the three benchmarks: WIG20 index, Allianz Akcji FIO mutual fund and Allianz OFE pension fund in the same period. Additionally an equally weighted portfolio of WIG20 stocks is constructed (WIG20eq), which serves as the fourth benchmark showing an alternative of passive investment in WIG20 stocks with equal weights. The equally weighted portfolio of WIG20 stocks is used instead of WIG20 index, since the data of stocks applied in the study is corrected by dividends which is not the case for WIG20 index. Thus, comparison of the models with equally weighted portfolio of WIG20 is more reliable than with WIG20 index.

Table 3. Basic statistics for the benchmarks in the period from January 1, 2007 to December 31, 2011

	<b>WIG20eq</b>	<b>WIG</b>	<b>Allianz OFE</b>	<b>Allianz Akcji FIO</b>
<b>ARC</b>	-9,13%	-5,89%	2,34%	-4,47%
<b>ASD</b>	27,47%	24,67%	7,15%	20,96%
<b>RR</b>	-0,33	-0,24	0,33	-0,21

\* Table 3 presents basic statistics for four benchmarks: WIG20eq, WIG, Allianz OFE and Allianz Akcji FIO in the period from January 1, 2007 to December 31, 2011.

Figure 1. The four benchmarks in the period from January 1, 2007 to December 31, 2011



\* Figure 1 presents the equity lines for four benchmarks: WIG20eq, WIG, Allianz OFE and Allianz Akcji FIO in the period from January 1, 2007 to December 31, 2011. The amount invested at the beginning of the period is 1 million Polish Zloty.

In Table 3 basic statistics for all the benchmarks are presented. In the studied period Allianz OFE pension fund was the best both in case of return and risk. It was the only benchmark with

positive return and positive value of return to risk ratio. For the other three benchmarks the return is negative and therefore the interpretation of return to risk ratio is ambiguous. Definitely the worst performance shows the WIG20eq with annual return of less than -9% and the highest annual standard deviation of more than 27%. WIG and Allianz Akcji FIO performed similar to each other with a slight advantage of the mutual fund.

Figure 1 presents the performance of the four benchmarks in the studied period. It is clearly visible that Allianz OFE pension fund is characterized by different behaviour than the other three benchmarks – during the crisis in 2008 it suffers small losses, whereas WIG, WIG20eq and Allianz Akcji FIO lose about half of their value in the worst moment of the crisis. Moreover, these three benchmarks behave similarly to each other.

All the calculations presented in the study were performed in the Microsoft Excel, version 2007.

#### 4. Results

In this section the results for all six models are described. Tables with aforementioned statistics as well as figures are presented in order to provide detailed view of models' performance. The models including transaction costs in the optimization process are compared with models without this constraint. Also models building portfolio from stocks are compared to models choosing from stocks and risk-free rate.

##### 4.1 Models with and without transaction costs as the optimization constraint

In the model specification, presented in section 3.2, the constraint (3.10) concerns transaction costs. Yu and Lee (2011) proposed the implementation of transaction costs in this way as an important improvement to the model. In this section the models including the constraint (3.10) are compared with models for which transaction costs are not considered as one of the optimization constraints. It is worth noting that in all models transaction costs are taken into account, but for some they are also considered as an optimization constraint.

Table 4. Basic statistics for models with transaction costs as the optimization constraint

	<b>M</b>	<b>MV</b>	<b>MS</b>	<b>MVK</b>	<b>MVS</b>	<b>MVSK</b>
<b>ARC</b>	-11,61%	-9,31%	-14,78%	-7,29%	-13,01%	-12,83%
<b>ASD</b>	29,03%	28,79%	29,57%	28,08%	29,31%	29,05%
<b>average t.c.</b>	0,21%	0,21%	0,27%	0,21%	0,27%	0,27%
<b>skewness<sup>6</sup></b>	-0,19	-0,18	-0,08	-0,28	-0,10	-0,11
<b>kurtosis<sup>7</sup></b>	2,09	2,11	1,95	2,26	1,99	1,78

\* Table 4 presents basic statistics for all the models in the period from January 1, 2007 to December 31, 2011. The portfolios of the models are reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 20% of portfolio value. For these models transaction costs are included in the optimization process as a constraint.

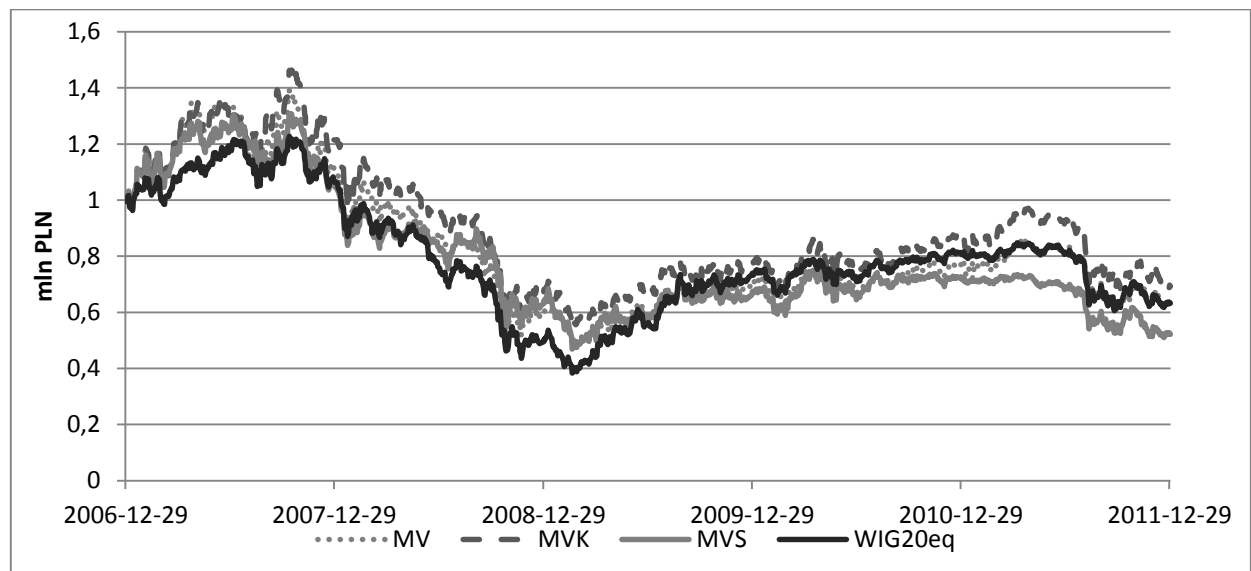
<sup>6</sup> Skewness is calculated according to the formula  $\frac{n}{(n-1)(n-2)} \sum \left( \frac{x_j - \bar{x}}{s} \right)^3$ , where n is the sample size, s standard deviation,  $x_j$  are the rates of returns and  $\bar{x}$  is the average rate of returns.

<sup>7</sup> Kurtosis is calculated according to the formula  $\frac{n(n+1)}{(n-1)(n-2)(n-3)} \sum \left( \frac{x_j - \bar{x}}{s} \right)^4 - \frac{3(n-1)^2}{(n-2)(n-3)}$ .

Table 4 presents basic statistics for all six models including transaction costs as an optimization constraint. Each model suffered significant losses in the studied period. The highest losses occurred for MS, whereas the lowest for MVK. It is worth noting that addition of skewness deteriorated results in case of every model – the MS, MVS and MVSK performed worse than M, MV and MVK respectively. The level of risk is similar for every model with about 29% annual standard deviation. The average transaction cost was above 0,2%, which means that on average more than 25% of the portfolio was replaced by each rebalancing. As far as skewness is concerned all the values are negative, which is probably a consequence of the large drawdown on the market during the world-wide crisis in 2008. The models including skewness as optimization constraint – MS, MVS, MVSK – obtained higher values of skewness than their counterparts without this constraint – M, MV, MVK. It shows that the addition of skewness allows influencing skewness of the portfolio distribution of returns, but the improvement is relatively small. The value of kurtosis higher than 0 informs that the distribution is leptokurtic with fatter tails than normal distribution and the phenomenon of leptokurtosis is present in distributions of returns of all the models. Comparing models with kurtosis as optimization constraint – MVK, MVSK – with their counterparts without this constraint – MV and MVS it cannot be stated that addition of kurtosis decreases kurtosis of portfolio returns. In case of MV model the kurtosis increased, whereas for MVS the kurtosis decreased when the additional constraint for kurtosis was imposed.

On Figure 2 the MV, MVK, MVS models and WIG20eq are presented. The equity lines of the portfolios are very similar to each other. The MVK – model with the highest ARC from models in Table 4, outperforms WIG20eq, while the MVS, the model with the lowest ARC, performs worse than WIG20eq. Based on the abovementioned results it cannot be stated that the models obtain good results or even outperform the benchmark. In the next part of this section it is checked whether the removal of transaction costs optimization constraint influences the results.

Figure 2. WIG20eq and the MV, MVK, MVS with transaction costs as an optimization constraint



\* Figure 2 presents equity lines for MV, MVK, MVS models and benchmark WIG20eq in the period from January 1, 2007 to December 31, 2011. The portfolios of the models are reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 20% of portfolio value. The amount invested at the beginning of the period is 1 million Polish Zloty. For these models transaction costs are included in the optimization process as a constraint.

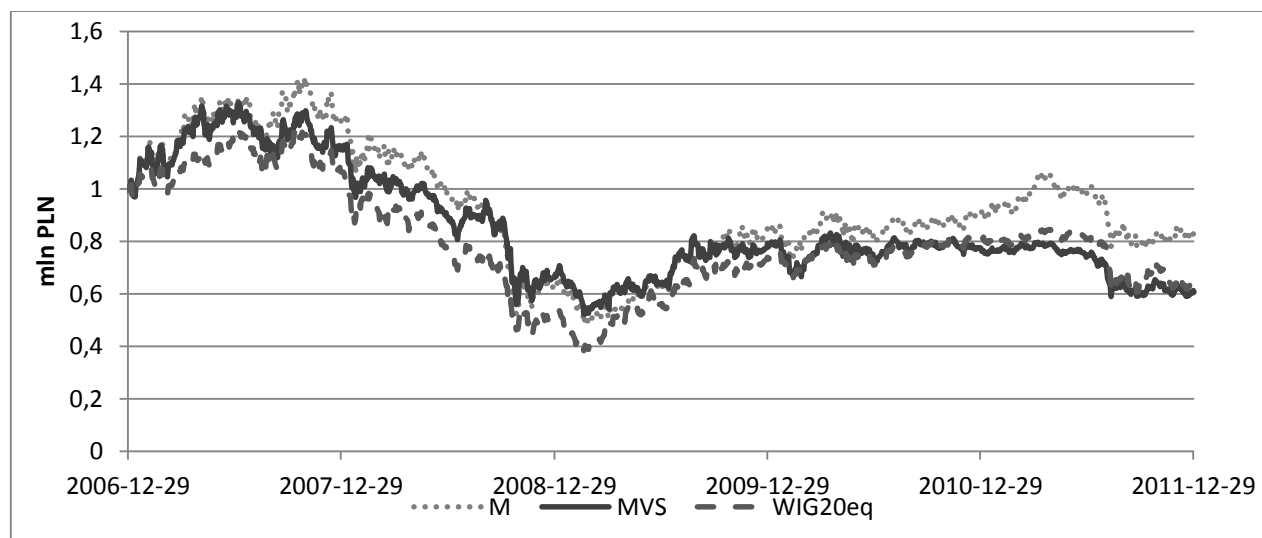
Table 5 presents basic statistics for models not including transaction costs in the optimization process. The annual return is negative but the results are much better than in case of models presented in Table 4. The volatility is about 29%, a very similar value in comparison to models from Table 4. This means that the inclusion of transaction costs constraint in the optimization process does not have any influence on the volatility of the models' results. The average transaction cost amounts to about 0,48% which is more than two times higher as in case of models from Table 4. This is consistent with expectations, since models that constrain transaction costs obtain much lower values of costs. Similarly to the results presented in Table 4 also for models from Table 5 the addition of skewness as an optimization constraint improves skewness of the portfolio rates of returns, which is visible by comparing M, MV, and MVK with MS, MVS and MVSK models. Also the addition of kurtosis allows decreasing its value in the portfolios distributions of returns – MV and MVS models have fatter tails than MVK and MVSK.

Table 5. Basic statistics for models without transaction costs as the optimization constraint

	<b>M</b>	<b>MV</b>	<b>MS</b>	<b>MVK</b>	<b>MVS</b>	<b>MVSK</b>
<b>ARC</b>	-3,75%	-6,50%	-8,90%	-4,78%	-9,94%	-7,66%
<b>ASD</b>	29,07%	28,78%	28,95%	28,21%	28,78%	28,11%
<b>average t. c.</b>	0,46%	0,48%	0,49%	0,48%	0,48%	0,49%
<b>skewness</b>	-0,13	-0,14	-0,04	-0,14	-0,04	0,00
<b>kurtosis</b>	2,06	2,26	2,18	2,08	2,17	2,00

\* Table 5 presents basic statistics for all the models in the period from January 1, 2007 to December 31, 2011. The portfolios of the models are reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 20% of portfolio value. For these models transaction costs are not included in the optimization process as a constraint.

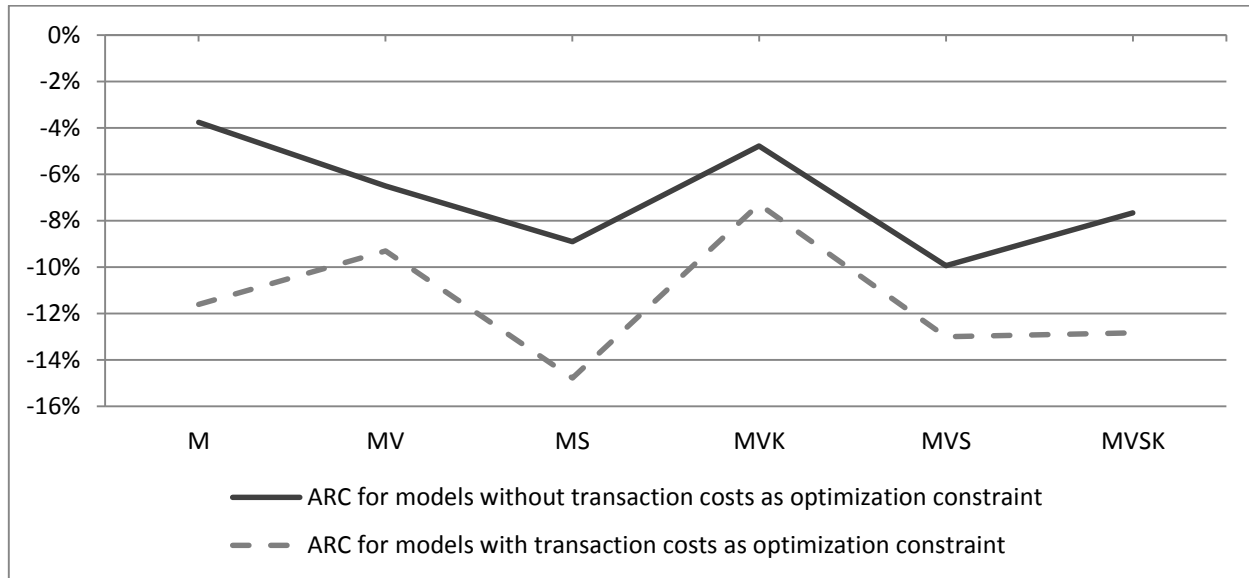
Figure 3. WIG20eq and the M, MVS without transaction costs as optimization constraint



\* Figure 3 presents equity lines for M, MVS models and benchmark WIG20eq in the period from January 1, 2007 to December 31, 2011. The portfolios of the models are reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 20% of portfolio value. The amount invested at the beginning of the period is 1 million Polish Zloty. For these models transaction costs are not included in the optimization process as a constraint.

Figure 3 shows performance of two models M, MVS and WIG20eq. During the first four years the models outperform the benchmark. At the end of the studied period MVS slightly underperforms in comparison to WIG20eq, whereas M obtains better results.

Figure 4. Comparison of ARC<sup>8</sup> for models with and without transaction costs as an optimization constraint



\* Figure 4 presents Annual Return Compounded for all the models in the period from January 1, 2007 to December 31, 2011. Models with and without transaction costs as an optimization constraint are presented. The portfolios of the models are reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 20% of portfolio value.

Figure 4 presents ARC for models with and without transaction costs as an optimization constraint. The Figure confirms the conclusions drawn from comparison of statistics from Table 4 and 5. All the models not having the transaction costs constraint achieve higher returns than their counterparts, which have this constraint. These results are surprising, because it was expected that models, for which transaction costs are constrained, should achieve higher returns due to lower costs. It turned out that indeed the models controlling transaction costs in optimization process bear lower costs, but it does not translate into higher returns. The cause of this situation may be the fact that the constraint (3.10) may heavily influence the choice of the optimal portfolio allowing only small changes in its composition. Thus, in situations where market conditions changed and the held portfolio was no longer optimal, the transaction costs constraint may allow in the reconstruction to obtain only suboptimal portfolio as far as the shape of the returns' distribution is concerned.

The results indicate that savings obtained by lower transaction costs in models controlling costs do not compensate the lost return that occurred due to additional transaction costs constraint. Therefore in the portfolio choice, the models without the constraint of transaction costs should be used, which is contrary to what Yu and Lee (2011) propose. Probably the inclusion of transaction costs might improve the results, but it should be done in different way, so that the other optimization constraints would not be so strongly affected.

<sup>8</sup> The ARC is presented on Figures in this paper and not Return to Risk ratio, because for many models the rate of returns are negative, which makes the interpretation of RR ambiguous – RR can be properly interpreted only for positive values.

## 4.2 Addition of risk-free asset to the models

In this subsection the addition of risk-free asset (WIBID3M) to the portfolio is analysed. It is supposed that risk-free asset, as a portfolio component of different characteristics from stocks, can add value to the portfolio performance and can help reduce losses during periods of downturns. Here only models without transaction costs as an optimization constraint are presented, since in previous section it was shown that they perform better from models controlling transaction costs.

Table 6. Basic statistics for all the models with risk-free asset and benchmarks

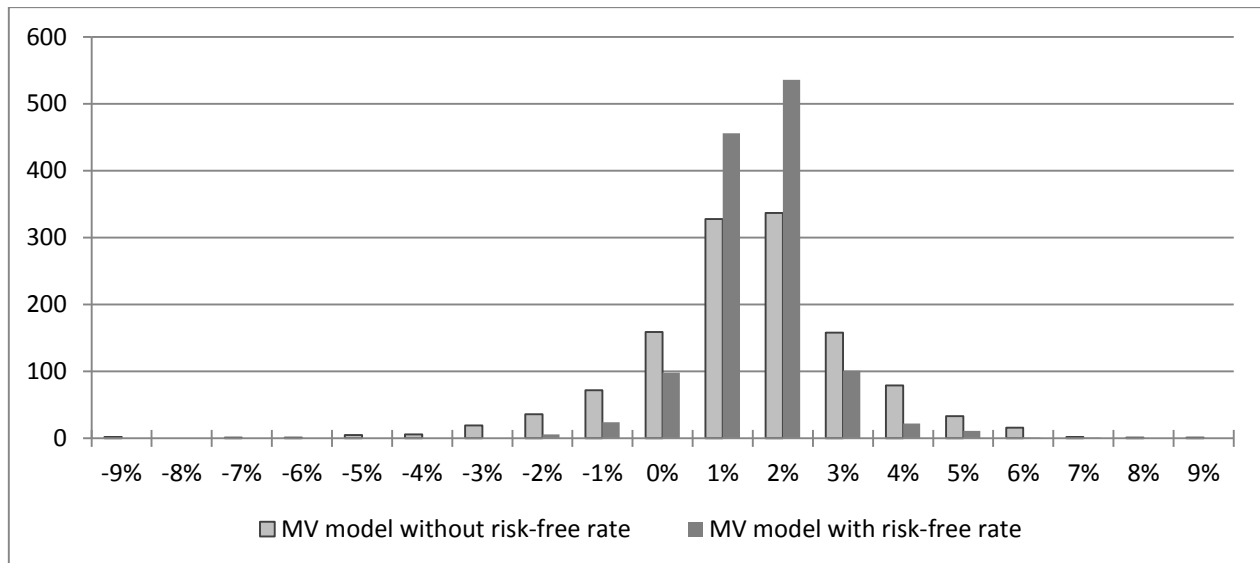
	M	MV	MS	MVK	MVS	MVSK	WIG20eq	WIG	Allianz OFE	Allianz Akcji FIO
<b>ARC</b>	8,41%	4,59%	3,01%	4,92%	3,10%	0,69%	-9,13%	-5,89%	2,34%	-4,47%
<b>ASD</b>	22,21%	14,92%	21,98%	14,94%	16,16%	16,34%	27,47%	24,67%	7,15%	20,96%
<b>RR</b>	0,38	0,31	0,14	0,33	0,19	0,04	-0,33	-0,24	0,33	-0,21
<b>skewness</b>	0,15	0,35	0,16	0,34	0,43	0,37	-0,25	-0,37	-0,46	-0,69
<b>kurtosis</b>	2,54	3,70	3,06	3,68	3,17	3,17	2,99	2,65	1,85	8,89

\* Table 6 presents basic statistics for all the models and benchmarks in the period from January 1, 2007 to December 31, 2011. The portfolios of the models are reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 20% of portfolio value and in risk-free rate is 100% of the portfolio value.

In Table 6 the statistics for all the models and benchmarks are shown. The annual return for all models is positive, which is a great result taking into account that the studied period includes the financial crisis. The models also perform well in comparison to the benchmarks. All of them beat WIG20eq, WIG and Allianz Akcji FIO significantly. Five models have higher rate of return than the best benchmark Allianz OFE. As far as both risk and return are concerned M, MV and MVK obtained similar results to Allianz OFE. Apart from much higher rate of return of the models with risk free rate in comparison to the models presented in the previous section, the addition of risk free rate allowed also to reduce significantly the volatility of the models' portfolio returns. The distributions of returns for all the benchmarks are negatively skewed and on the contrary the distributions of returns of each models' portfolios are characterized by positive skewness. Comparing it with the results from Table 5 it can be stated that the addition of risk-free rate improved skewness of the portfolios significantly. As far as kurtosis is concerned all models and benchmarks have positive kurtosis showing that leptokurtosis is present in each presented investment strategy.

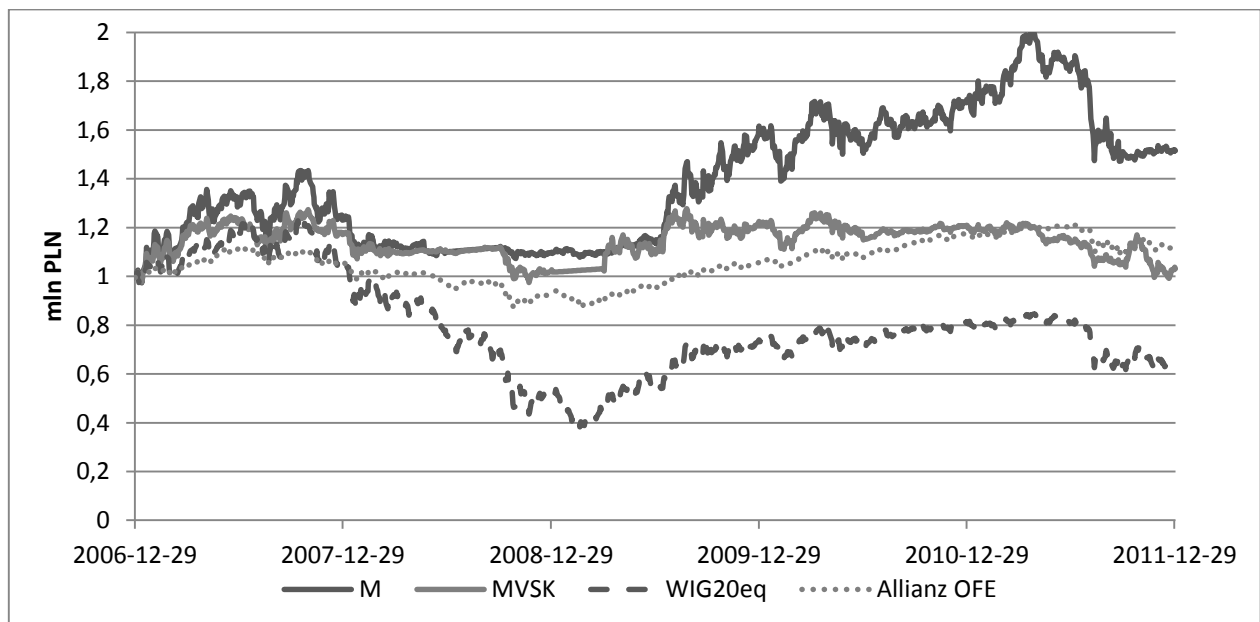
Figure 5 presents the frequency of the portfolios daily returns for two variants of MV model: one composed only of stocks and the second consisting of stocks and risk-free rate. The analysis of the Figure allows to state that the model with addition of risk-free rate is characterized by higher mean, much lower dispersion around the mean and higher skewness, resulting probably from much shorter left tail of the distribution, when it is compared to the model for which the investment in risk-free rate is not allowed. This Figure confirms the conclusions drawn from analysis of Tables 5 and 6 and visually shows the benefits of incorporation of risk-free asset into portfolio selection process.

Figure 5. The frequency of the daily rates of returns for two cases of the MV model – with and without risk-free rate



\* Figure 5 presents a histogram for rates of returns for MV models in the period from January 1, 2007 to December 31, 2011. Two variants of MV model are presented: one that allowed investment only in WIG20 stocks, and second one that enabled investment in WIG20 stocks and risk-free rate (WIBID 3M). The portfolios of the models are reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 20% of portfolio value and in risk-free rate is 100% of the portfolio value.

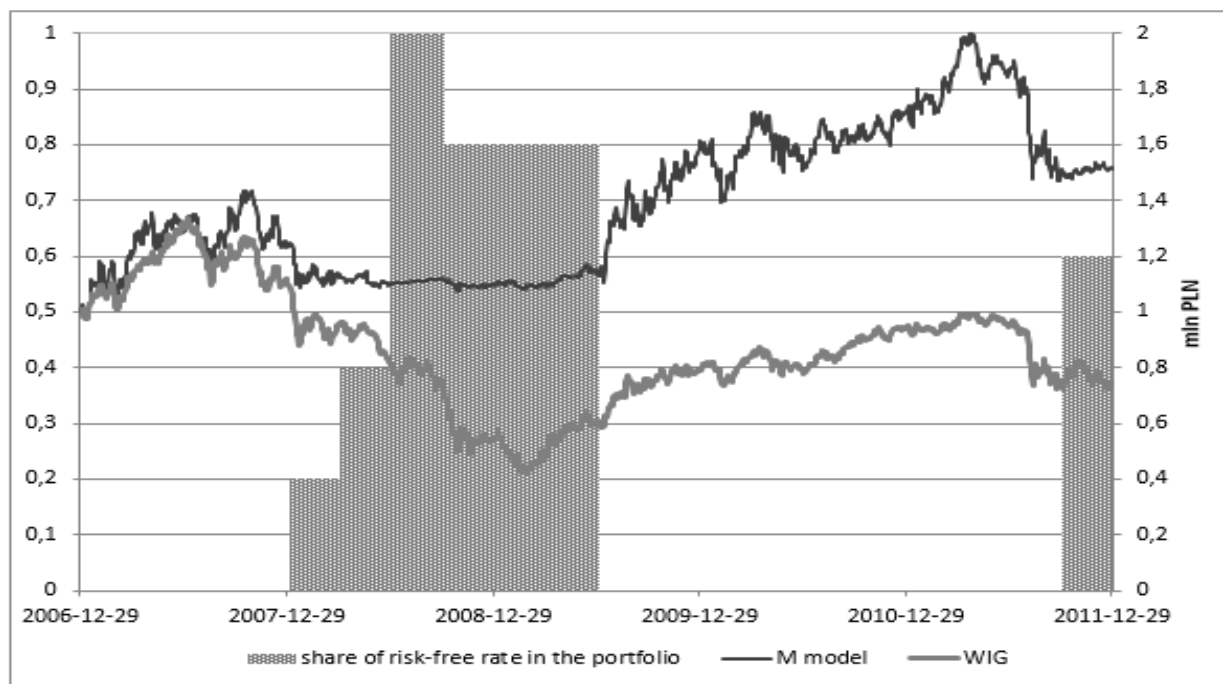
Figure 6. M, MVSK models with risk-free rate compared with Allianz OFE and WIG20eq



\* Figure 6 presents equity lines for M, MVSK models and two benchmarks: WIG20eq, Allianz OFE in the period from January 1, 2007 to December 31, 2011. The portfolios of the models are reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 20% of portfolio value and in risk-free rate is 100% of the portfolio value. The amount invested at the beginning of the period is 1 million Polish Zloty.

Figure 6 presents the equity lines of benchmarks and models with the highest and lowest annual rate of return – Allianz OFE, WIG20eq, M and MVSK models respectively. The models outperform the WIG20eq significantly. The worst of the models in case of return – MVSK – performs very similar to the best benchmark Allianz OFE. Summing up the information from Table 6 and Figure 6 it can be concluded that the addition of risk-free asset improves the results significantly. Not only volatility is diminished but also the return is improved. It is worth noting that the models with risk free asset protect the invested capital, which is an important quality especially in periods of large downturns like the one that occurred in 2008.

Figure 7. M model, WIG and share of risk-free rate in M model portfolio



\* Figure 7 presents equity lines for M model and WIG in the period from January 1, 2007 to December 31, 2011. The grey columns represent the share of risk-free rate in M model portfolio in a given quarter. The share is measured on the left vertical axis. The portfolio of the model is reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 20% of portfolio value and in risk-free rate is 100% of the portfolio value. The amount invested at the beginning of the period is 1 million Polish Zloty.

Figure 7 shows the performance of the M model in comparison to WIG and additionally the share of risk-free asset in the M model's portfolio in the studied period. The model did not choose the risk-free asset in 2007 at all. But in the 2008, when the situation on the financial market was getting worse, the share of risk-free asset was gradually increasing. Thus, in 2008 and in the first half of 2009 the model invested large part of the portfolio value in the risk free asset, which proved to be a great strategy against losses that occurred on markets in that time. When the situation on the stock exchange was becoming better the model stopped investing in risk-free asset, which allowed making a high profit. The model invested also in the risk-free rate in the last quarter of 2011, which was a reaction to large losses on the market in third quarter of 2011. But this time the model did not foresee the downturn of August 2011. The reason for this may be the fact that the model is rebalanced only quarterly and therefore cannot react sufficiently fast to

sudden changes on the market. It is also possible that the model did not react strong enough to the losses, because its parameters are estimated on 6 month historical data. With shorter periods of historical data, e.g. 3 months, the model would react stronger to recent events on the market.

## 5. Sensitivity analysis

In this section the influence of changes in the research assumptions on the results is presented. Firstly, the change of the upper bound of a position in one security from 20% to 10% and 40% is examined. Then the shift of the starting point of portfolio reconstruction from the beginning of the quarter to the middle of the quarter is investigated. Subsequently, monthly portfolio rebalancing is compared to quarterly rebalancing. Then the influence of transaction costs on the results is studied and at the end the optimal models are described. The sensitivity analysis is conducted for all the six models without transaction costs as an optimization constraint and with risk-free asset. Therefore the models that performed the best according to section 4 are analysed.

### 5.1 Change of upper bound of a position in one security

Table 7 presents basic statistics for all the models with upper bound of a position in one security shifted from 20% to 10% and 40%. This shift concerns only stocks, since in case of risk free rate always the investment up to 100% is possible. Generally the decrease in the upper bound of a position improves the results for all models, except from M model for which the highest RR is for upper bound of 20%.

Table 7. Basic statistics for models with upper bound of a position in one security of 10%, 20% and 40%

		M	MV	MS	MVK	MVS	MVSK
<b>upper bound of a position of 0,1</b>	<b>ARC</b>	5,03%	7,53%	2,44%	7,03%	5,69%	1,57%
	<b>ASD</b>	19,17%	14,81%	18,06%	14,82%	14,99%	15,81%
	<b>RR</b>	0,26	0,51	0,14	0,47	0,38	0,10
<b>upper bound of a position of 0,2</b>	<b>ARC</b>	8,41%	4,59%	3,01%	4,92%	3,10%	0,69%
	<b>ASD</b>	22,21%	14,92%	21,98%	14,94%	16,16%	16,34%
	<b>RR</b>	0,38	0,31	0,14	0,33	0,19	0,04
<b>upper bound of a position of 0,4</b>	<b>ARC</b>	0,46%	2,83%	-2,40%	2,87%	0,30%	-1,19%
	<b>ASD</b>	25,45%	14,74%	24,65%	14,74%	15,82%	16,06%
	<b>RR</b>	0,02	0,19	-0,10	0,19	0,02	-0,07

\* Table 7 presents basic statistics for all the models in the period from January 1, 2007 to December 31, 2011. Three variants of the models are distinguished for which maximum investment in stocks of one company is 40%, 20% and 10%. The maximum investment in risk-free rate is 100% of the portfolio value. The portfolios of the models are reconstructed at the beginning of every quarter, based on half-year historical data. The transaction cost is 0,4% of the transaction value.

The analysis of Table 7 allows to state that in case of rate of return the models with upper bound of a position in one security of 20% outperform models with higher upper bound of 40% and models with upper bound of 10% outperform models with upper bound of 20% in 4 out of 6

cases. This shows that it is worth to impose a restriction of a maximal share of one security in the portfolio and that the results are highly affected by the level of the imposed bounds. Too high value of the boundary deteriorated the results. It allowed a higher concentration in a smaller number of securities, which proved not to be a good investment strategy. Additional conclusion, which can be drawn from the analysis of Table 7, is that models with skewness have lower rates of return than their counterparts without skewness. Models M, MV, MVK achieve better results than MS, MVS and MVSK respectively.

## 5.2 Change of the portfolio reconstruction moment to the middle of the quarter

In this section the influence of the moment of portfolio reconstruction on the stability of the results is examined. So far all the results were presented for models with portfolio reconstruction occurring at the beginning of each quarter during the studied period. Now the moment of portfolio reconstruction is shifted to the middle of the quarter.

Table 8. Basic statistics for models with different moments of portfolio reconstruction

		M	MV	MS	MVK	MVS	MVSK
<b>Beginning of the quarter rebalancing</b>	<b>ARC</b>	8,41%	4,59%	3,01%	4,92%	3,10%	0,69%
	<b>ASD</b>	22,21%	14,92%	21,98%	14,94%	16,16%	16,34%
	<b>RR</b>	0,38	0,31	0,14	0,33	0,19	0,04
<b>Middle of the quarter rebalancing</b>	<b>ARC</b>	4,08%	9,12%	-0,29%	9,33%	0,68%	2,45%
	<b>ASD</b>	22,44%	14,89%	20,71%	15,01%	16,43%	16,77%
	<b>RR</b>	0,18	0,61	-0,01	0,62	0,04	0,15

\* Table 8 presents basic statistics for all the models in the period from February 14, 2007 to February 14, 2012. Two variants of the models are distinguished for which the moment of portfolio reconstruction is at the beginning of the quarter and in the middle of the quarter. The maximum investment in stocks of one company is 20% of portfolio value and in risk-free rate is 100% of the portfolio value. The portfolios of the models are reconstructed based on half-year historical data. The transaction cost is 0,4% of the transaction value.

Table 8 shows the results both for models with the portfolio reconstruction moment at the beginning of the quarter and in the middle of the quarter. As far as the volatility is concerned, the moment of portfolio reconstruction almost does not influence it. The changes in volatility are negligible. But the rate of return and likewise return to risk ratio change significantly. Nevertheless for both variants of the portfolio reconstruction moments the three best models are the same – M, MV and MVK. But the hierarchy changes: with beginning of the quarter rebalancing the M model is the best one, whereas with middle of the quarter rebalancing the MVK is the best one.

This part of sensitivity analysis proves that there is any model that outperforms other ones irrespective of assumptions made. The shift of the portfolio reconstruction moment changes the hierarchy of the models. The analysis also shows how important the assumptions are and how sensitive the models can be for slight changes in them.

### 5.3 Monthly portfolio reconstruction

In this section the frequency of portfolio reconstruction is changed. So far for all the presented models it was assumed that composition of the portfolio is subject to change once a quarter. On the one hand this approach allowed to bear quite a low transaction costs, when the portfolio is reconstructed only four times a year, but on the other hand a quarter is quite a long time and when sudden changes occur on the market, the model reacts with a large delay. In this section it is examined whether monthly rebalancing of the portfolio, instead of quarterly, can improve the results.

Table 9. Basic statistics for models with different frequency of portfolio reconstruction

		<b>M</b>	<b>MV</b>	<b>MS</b>	<b>MVK</b>	<b>MVS</b>	<b>MVSK</b>
<b>quarterly portfolio reconstruction</b>	<b>ARC</b>	8,41%	4,59%	3,01%	4,92%	3,10%	0,69%
	<b>ASD</b>	22,21%	14,92%	21,98%	14,94%	16,16%	16,34%
	<b>RR</b>	0,38	0,31	0,14	0,33	0,19	0,04
<b>monthly portfolio reconstruction</b>	<b>ARC</b>	7,48%	8,32%	3,75%	7,70%	4,83%	2,22%
	<b>ASD</b>	22,73%	14,94%	23,05%	14,99%	16,40%	16,43%
	<b>RR</b>	0,33	0,56	0,16	0,51	0,29	0,14

\* Table 9 presents basic statistics for all the models in the period from January 1, 2007 to December 31, 2011. Two variants of models are distinguished for which the portfolios are rebalanced monthly and quarterly. The maximum investment in stocks of one company is 20% of portfolio value and in risk-free rate is 100% of the portfolio value. The portfolios of the models are reconstructed based on half-year historical data. The transaction cost is 0,4% of the transaction value.

Table 9 shows basic statistics for models with both monthly and quarterly portfolio rebalancing. For five out of six models the rate of returns and return to risk ratios are higher when the composition of the portfolio is changed once a month. The largest improvement is for MV and MVK models, for which the annual rate of return increases from 4,59% and 4,92% to 8,32% and 7,7% respectively. The volatility for most of the models does not change significantly.

Although monthly rebalancing does not outperform quarterly rebalancing for all the models, it should be stressed that in case of monthly portfolio reconstruction the total transaction costs should be much higher. The influence of transaction costs on the results is examined in the next section.

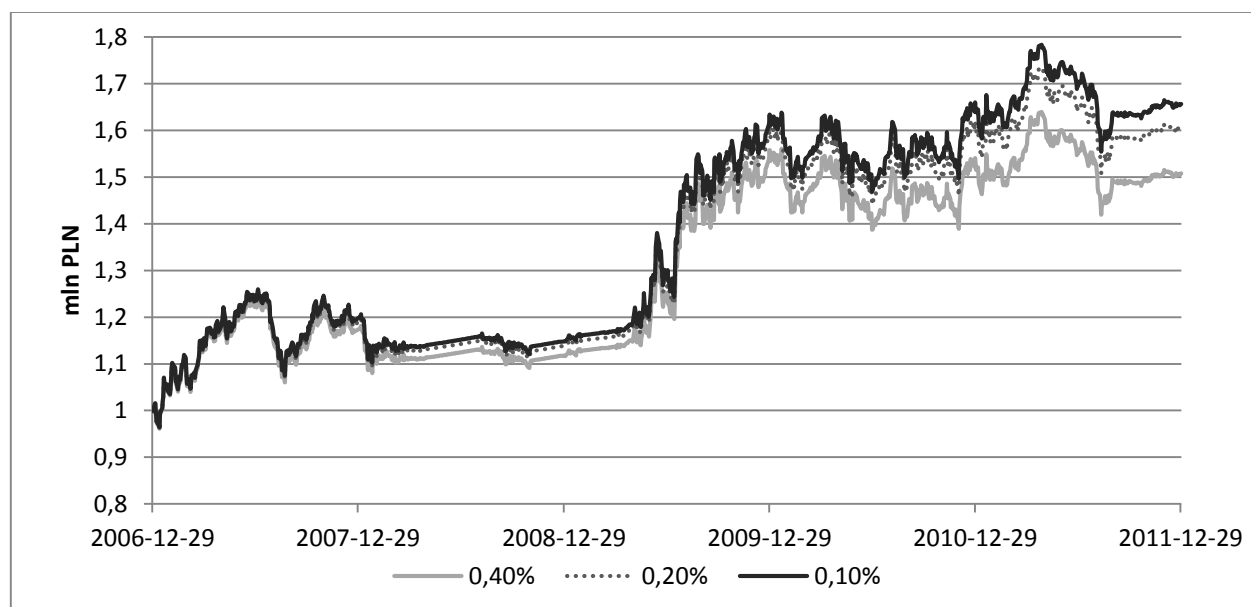
### 5.4 The influence of transaction costs on models' performance

In this section the influence of transaction costs on the models' performance is analysed. In the whole research the assumed level of transaction costs is 0,4% of the transaction value. This size of costs is close to reality in case of individual investors. But institutional investors executing large deals can pay much lower costs in proportion to the transaction value. Thus, in this section the sensitivity of the results to the level of transaction costs is checked. Three levels of costs are compared: 0,1%, 0,2% and 0,4%. The results are presented for the best model with quarterly and monthly reconstruction when it comes to value of return to risk ratio, i.e. M and MV models respectively. The comparison of models with different frequency of portfolio reconstruction

might be interesting due to the higher transaction costs expected for model with more frequent portfolio reconstruction.

Figure 8 shows equity lines of MV model rebalanced monthly for three levels of transaction costs. It is visible that costs accumulate in time. During the first two years of the investment period the difference between the portfolio values with different costs are almost negligible, but at the end of the investment the impact of costs becomes significant.

Figure 8. MV model rebalanced monthly with transaction costs of 0,4%, 0,2% and 0,1%



\* Figure 8 presents equity lines for MV model in the period from January 1, 2007 to December 31, 2011. Models with transaction costs of 0,40%, 0,20% and 0,10% of the transaction value are presented. The portfolios of the models are reconstructed at the beginning of every month, based on half-year historical data. The maximum investment in stocks of one company is 20% of portfolio value and in risk-free rate is 100% of the portfolio value. The amount invested at the beginning of the period is 1 million Polish Zloty.

Table 10. Basic statistics for M model rebalanced quarterly and MV model rebalanced monthly for transaction costs levels of 0,4%, 0,2% and 0,1%

transaction costs	M rebalanced quarterly			MV rebalanced monthly		
	0,4	0,2	0,1	0,4	0,2	0,1
ARC	8,41%	9,17%	9,55%	8,32%	9,54%	10,14%
ASD	22,21%	22,20%	22,20%	14,94%	14,93%	14,93%
RR	0,38	0,41	0,43	0,56	0,64	0,68

\* Table 10 presents basic statistics for M model rebalanced quarterly and MV model rebalanced monthly in the period from January 1, 2007 to December 31, 2011. The statistics are presented for three variants of transaction costs for each model: 0,40%, 0,20% and 0,10%. The portfolios of the models are reconstructed based on half-year historical data. The maximum investment in stocks of one company is 20% of portfolio value and in risk-free rate is 100% of the portfolio value.

Table 10 presents statistics for both models. The change in transaction costs influences highly annual return and has negligible impact on volatility. As it was expected, the change of transaction costs impacts more the results of model with monthly rebalancing than model with quarterly rebalancing. The reduction of transaction costs from 0,4% to 0,1% improves the annual return for MV by almost 2 percentage points and for M model by slightly more than 1 percentage point. After five years of investment it translates into a high difference of portfolio values for different values of transaction costs.

## 5.5 The choice of optimal models

In this section the best models according to results from previous sections are presented. In Section 4 it was shown that the constraint of transaction costs shouldn't be included in the process of optimization. Additionally it occurred that it is worth including a risk-free rate in a set of investable securities apart from stocks. From the analysis conducted in sections 5.1 and 5.3 it follows that the best results obtain models with 10% upper bound of a position in one security and in case of monthly portfolio reconstruction. In this section the models fulfilling all the abovementioned criteria are analysed.

Table 11. Basic statistics for models with monthly portfolio reconstruction and upper bounds of 10% and 20%

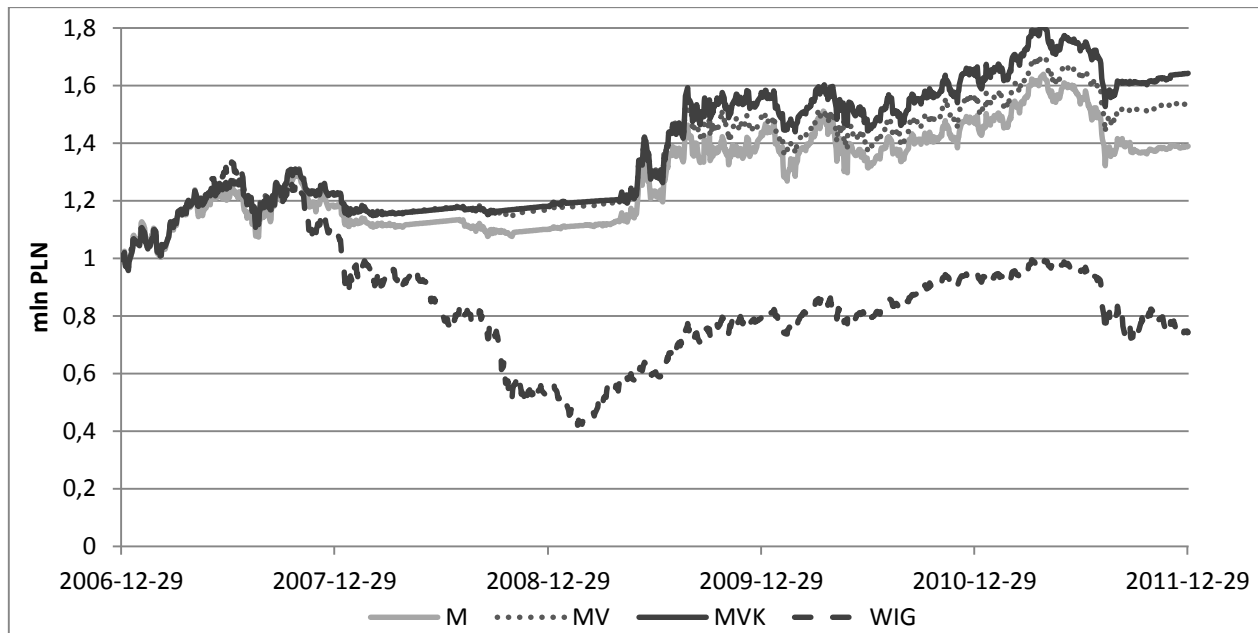
		M	MV	MS	MVK	MVS	MVSK
<b>monthly portfolio reconstruction with upper bound of 0,1</b>	<b>ARC</b>	6,67%	8,70%	5,19%	10,03%	6,25%	4,10%
	<b>ASD</b>	18,16%	15,41%	17,01%	15,23%	15,24%	15,84%
	<b>RR</b>	0,37	0,56	0,30	0,66	0,41	0,26
<b>monthly portfolio reconstruction with upper bound of 0,2</b>	<b>ARC</b>	7,48%	8,32%	3,75%	7,70%	4,83%	2,22%
	<b>ASD</b>	22,73%	14,94%	23,05%	14,99%	16,40%	16,43%
	<b>RR</b>	0,33	0,56	0,16	0,51	0,29	0,14

\* Table 11 presents basic statistics for all the models in the period from January 1, 2007 to December 31, 2011. Two variants of models are distinguished for which the maximum investment in stocks of one company is 10% and 20% of the portfolio value. The maximum investment in risk-free rate is 100% of the portfolio value. The portfolios of the models are reconstructed at the beginning of every month, based on half-year historical data. The transaction cost is 0,4% of the transaction value.

In Table 11 the models with monthly portfolio reconstruction and upper bound of a position of 10% are compared to the best models obtained in the previous sections, which are models reconstructed monthly with upper bound of 20%. As it was expected, the combination of monthly portfolio reconstruction and 10% upper bound allows improving the results. In case of 5 out of 6 models the annual rate of return is higher for models with 10% upper bounds. Only for the M model the ARC deteriorates slightly, but it is accompanied by improvement in ASD, which finally translated into better return to risk ratio (RR).

Figure 9 shows performance of three models with highest ARC and WIG. The models behave similarly. All of them are characterized by an uptrend and relatively low volatility, which makes them an attractive investment tool. They beat the WIG significantly.

Figure 9. WIG and M, MV, MVK models reconstructed monthly with upper bound of a position in one security of 10%



\* Figure 9 presents equity lines for M, MV, MVK models and WIG in the period from January 1, 2007 to December 31, 2011. The models are reconstructed at the beginning of every month, based on half-year historical data. The transaction cost is 0,4% of the transaction value. The maximum investment in stocks of one company is 10% of portfolio value and in risk-free rate is 100% of the portfolio value. The amount invested at the beginning of the period is 1 million Polish Zloty.

As far as the whole research is concerned a few very interesting conclusions can be drawn:

- The inclusion of transaction costs as one of optimization constraints deteriorated results. This constraint had high and undesirable influence on return distribution. Although the transaction costs were lower, the models performed worse than in case without constraint of transaction costs.
- The addition of risk-free asset to the set of investable securities improved the results. It allowed the models to save capital when large losses occurred in the market. Thus, when building a portfolio selection model it is worth to take into account both risky assets and a risk-free asset. It allows to avoid losses during market downturns and to make high profits in the bull market.
- It is better to reconstruct a portfolio monthly rather than quarterly. Although the transaction costs are then higher it allows the models to better react to changing market conditions and to make higher profits.
- It is worth to impose constraints of a maximum share of stocks of one company in the portfolio. The upper bound of 10% allowed to diversify the portfolio and thus lower the volatility and to avoid a too high concentration in a few stocks.

The results of this research present criteria that should be taken into account when a portfolio selection models are built. It was also showed that the models can be very sensitive to the assumptions made. Although finding and parameterizing a good portfolio selection model is not an easy task it might be worthwhile, because a good model can beat the market as it was shown in this research.

## 6. Conclusions

The presented study shows that with the application of portfolio selection models, which base only on historical data of securities returns, it is possible to obtain better than average results, when compared to the benchmarks. All the six models, which allowed investment in WIG20 stocks and risk-free rate and do not include transaction costs constraint in the model, obtained much better results when it comes to return, return to risk ratio and skewness in comparison to WIG, Allianz Akcji FIO and equally weighted portfolio of WIG20 stocks. Additionally three of the models – M, MV, MVK, achieved risk to return ratio similar to that of Allianz OFE. Thus, the first research hypothesis is confirmed. As far as the analysis of moments higher than variance is concerned, the results are ambiguous. The inclusion of skewness as an optimization constraint indeed increased the skewness of the portfolio, but at the expense of a lower rate of return. The kurtosis constraint decreased kurtosis of the portfolio in majority of the cases, thus diminishing the probability of extreme events, but it did not work for all the models.

Another important factor that highly influenced results were transaction costs. Contrary to what Yu and Lee (2011) proposed, it turned out that the inclusion of costs in the optimization as a constraint distorts the results – the models without this constraint performed much better. Also the level of transaction costs has high influence on results and its impact rises when the portfolio is more frequently reconstructed. Moreover, the addition of risk-free rate to the pool of investable assets improved the results of all the models significantly. It allowed the models to switch to safe investment and to save the invested capital, when large losses occurred on the stock market. Additionally, the extensive sensitivity analysis proved that the performance of the models highly depends on the assumptions made. The results differed when the frequency and moment of the portfolio reconstruction was changed and when the maximum bounds of investment in stocks were altered. Nevertheless, in majority of the cases the M, MV and MVK models obtained the best results.

This research showed a small part of possibilities of multi-objective models in portfolio selection. Some of the models obtained extraordinary results, showing the potential of application of quantitative methods to solve such important real-life problems like the choice of investment assets to the portfolio. Although different models were analysed, there is still a lot of space for further research. A very interesting improvement might be an addition of other optimization criteria, e.g. the use of semi-variance or mean-absolute-deviation as a risk measure instead of variance. The application of the models on different markets and to portfolios composed of diverse assets like for example futures, with the possibility of short selling, might be an interesting alternative. Furthermore, the change of the estimation period for the input variables from half a year to shorter periods, or the exponential weighting of the observations to allow models to faster react to changes in the market, can be analysed. Also the implementation of dynamic reconstruction of the portfolios allowing reaction to changes in market conditions instead of reconstruction every fixed period seems to be worth considering.

## References

### Books and articles

Arditti F. D., 1971. *Another Look at Mutual Fund Performance*. The Journal of Financial and Quantitative Analysis 6, 909-912.

Chunhachinda P., Dandapani K., Hamid S., Prakash A. J., 1997. *Portfolio selection and skewness: Evidence from international stock markets*. Journal of Banking & Finance 21, 143-167.

Culp C. L., 2001. *The Risk Management Process: Business Strategy and Tactics*. John Wiley and Sons, New York.

Feibel B. J., 2003. *Investment performance measurement*. John Wiley and Sons, Hoboken.

Jean W. H., 1971. *The Extension of Portfolio Analysis to Three or More Parameters*. The Journal of Financial and Quantitative Analysis 6, 505-515.

Joro T., Na P., 2006. *Portfolio performance evaluation in a mean-variance-skewness framework*. European Journal of Operational Research 175, 446-461.

Konno H., Shirakawa H., Yamazaki H., 1993. *A mean-absolute deviation-skewness portfolio optimization model*. Annals of Operations Research 45, 205-220.

Konno H., Suzuki K., 1995. *A mean-variance-skewness portfolio optimization model*. Journal of the Operations Research Society of Japan 38, 173-187.

Lai K. K., Yu L., Wang S., 2006. *Mean-Variance-Skewness-Kurtosis-based Portfolio Optimization*. First International Multi-Symposiums on Computer and Computational Sciences 2, 292-297.

Lai T.-Y., 1991. *Portfolio Selection with Skewness: A Multiple-Objective Approach*. Review of Quantitative Finance and Accounting 1, 293-305.

Lee E. S., Li R. J., 1993. *Fuzzy multiple objective programming and compromise programming with Pareto optimum*. Fuzzy Sets and Systems 53, 275-288.

Markowitz H., 1952. *Portfolio Selection*. The Journal of Finance 7, 77-91.

Prakash A. J., Chang C.-H., Pactwa T. E., 2003. *Selecting a portfolio with skewness: Recent evidence from US, European, and Latin American equity markets*. Journal of Banking & Finance 27, 1375-1390.

Ryoo H. S., 2007. *A Compact Mean-Variance-Skewness Model for Large-Scale Portfolio Optimization and Its Application to the NYSE Market*. The Journal of the Operational Research Society 58, 505-515.

Samuelson P., 1970. *The Fundamental Approximation Theorem of Portfolio Analysis in terms of Means, Variances and Higher Moments*. The Review of Economic Studies 37, 537-542.

Scott R. C., Horvath P. A., 1980. *On the Direction of Preference for Moments of Higher Order than the Variance*. The Journal of Finance 35, 915-919.

Tang G. Y. N., Shum W. C., 2003. *The relationships between unsystematic risk, skewness and stock returns during up and down markets*. International Business Review 12, 523-541.

Wang S., Xia S., 2002. *Portfolio selection and asset pricing*. Springer, Berlin.

Yu J.-R., Lee W.-Y., 2011. *Portfolio rebalancing model using multiple criteria*. European Journal of Operational Research 209, 166-175.

Zimmermann H. J., 1978. *Fuzzy programming and linear programming with several objective functions*. Fuzzy Sets and Systems 1, 45-55.

### **Internet sites**

[www.wyborcza.biz](http://www.wyborcza.biz)

[www.gpw.pl](http://www.gpw.pl)



FACULTY OF ECONOMIC SCIENCES  
UNIVERSITY OF WARSAW  
44/50 DŁUGA ST.  
00-241 WARSAW  
[WWW.WNE.UW.EDU.PL](http://WWW.WNE.UW.EDU.PL)