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Generalisation of the gross flows model

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Abstract: In this study, we generalise Shimer's (2012) gross ow model in three directions using the Perron-Frobenius theorem. While the seminal work of Shimer (2012) analysed a labour market model with dynamic gross flows, three states of the economy, and stable population, our generalised model allows for considering more than three states of the economy, allows the size of the economy to change over time, and generalises the process of calculating steady states from empirical data. Our model provides a theoretical background for similar dynamic problems.

Keywords: dynamic market model, Perron-Frobenius theorem, worker flows, labour market status

JEL codes: C62, J21, J63

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1. Introduction

We observe numerous instances of transitions between states in everyday life; people change their family status, travel between geographical locations, or change their labour market status. In this context, Shimer (2012) proposed a labour market model with dynamic gross flows to study steady states based on the transitions between the three states of employment, unemployment, and inactivity. His model is based on the following two assumptions. First, workers do not enter or exit the labour force but simply transit between the states of the economy. Second, all workers are ex-ante identical; in particular, they possess the same probability of transitioning from a given state to another during any period of time.

However, Shimer's (2012) seminal model does not consider the occurrence of changes in the population at the labour market. Specifically, it assumes that the inflow or outflow of workers does not take place over time. However, this is contradictory to the data observed for economies exhibiting substantial cyclic fluctuations in the population, or structural differences between entries of new workers and the exits of workers in the case of retirement. Additionally, the number of young workers entering the labour market to initiate their professional career and older people exiting it in order to retire is subject to an unceasing evolution.

A common element in such flows lies in the fact that they can be described using the Perron-Frobenius theorem. Accordingly, we modify the equations to describe the flow of persons between the labour market states, allowing the size of the population to change over time. This flexible framework allows us to solve two problems related to the stable flow rates and stable population size in Shimers (2012) model.

We extend Shimer's (2012) gross flows model in three directions. First, we generalise the model by considering $n \geq 3$ potential states of the economy. Second, we relax the assumption of flow dynamics, following a Poisson arrival process. This assumption was used by Shimer (2012) to calculate the flows using an empirical data. Third, we allow the size of the economy to change over time as a result of the inflow of new workers in the labour market and outflow of current workers from the labour market. The first extension allows us to analyse the employment in different economic sectors within the same general theoretical framework. The second extension simplifies steady-state calculation and the third extensions allows to incorporate fluctuations in the economy size. Accordingly, we provide a general solution for the existence and the uniqueness of steady-state.

Following the Perron-Frobenius theorem and related concepts, we present a generalised mathematical framework that is applicable beyond the labour market for the analyses of fluctuations in the flows between states. We build our proofs based on the Perron-Frobenius theorem, which has been widely used in various fields of economics, such as asset pricing (Hansen and Scheikman, 2009; LeGrand, 2019), econometrics (Chen et al., 2014), spatial econometrics (Matrellosio, 2012), and network effects (Mastrobuoni, 2015).

2. Model extensions

First, we generalise Shimer's (2012) 3-dimensional model to an n-dimensional one ($n \geq 3$). At a given time t ($t \geq 0$), the state of the economy is formally described by an n-dimensional vector $v(t)$, such that:

$$\forall_{i \leq n} v_i(t) \geq 0 \ \wedge \ \sum_{i=1}^{N} v_i(t) = N,$$

where $N > 0$ denotes the constant size of the economy. The set of all potential states of the economy is denoted by S, that is, $S = \{v(t) : \sum_{i=1}^{n} v_i(t) = N, v_i(t) \geq 0\}$. Note that S is a convex and compact simplex in the n-dimensional Euclidean space spanned by the vertical vectors $(0, \dots, 0, N, 0, \dots, 0)'$. When $n = 3$ dimensions, S is a triangle. Here, $Int(S)$ denotes the interior of S, that is, $Int(S) = \{v(t) : \sum_{i=1}^{n} v_i(t) = N, v_i(t) > 0\}$. The evolution of the economy is deterministic and described by the system of $n \times n$ flow matrices $\{\Lambda(s,t) : t > s \geq 0\}$, such that $v(t) = \Lambda(s,t)v(s)$, for all $t > s \geq 0$.

The system of flow matrices satisfies the consistency condition, that is, for any t_1, t_2 and t_3, such that $t_3 > t_2 > t_1 \geq 0$, the following condition holds: $\Lambda(t_1, t_2)\Lambda(t_2, t_3) = \Lambda(t_1, t_3)$.

We assume that all matrices are stochastic with positive elements, that is, for all $t > s \geq 0$:

1. $\forall i, j \leq n, \Lambda_{ij}(s, t) > 0$, where i denotes a row and j denotes a column;
2. $\forall j \leq n$, $\sum_{i=1}^{n} \Lambda_{ij}(s, t) = 1$, that is, the sum of coefficients in each column equals to 1.

Both the above conditions jointly ensure that if $v(s)$ is a state of the economy, $v(t) = \Lambda(s, t)v(s)$ is also a valid state of the economy; therefore, if $v(s) \in S$, then $\Lambda(s, t)v(s) \in S$. The first condition implies that for a time interval $[s, t]$, the flow between the elementary states i and j $(i, j \leq n)$ is non-zero. Moreover, the first condition ensures that for any $t > s \geq 0$, $\Lambda(s, t)v(s) \in Int(S)$, that is, $\forall t > 0, v(t)$ is in the interior of S. Let us now fix $t > s \geq 0$, and denote the flow matrix $\Lambda(s, t)$ by Λ. We state that the economy is in a steady state $v^\star$, if $v^\star = \Lambda v^\star$, such that at time t, the economy returns to the same state as it was at time s. The following proposition ensures the existence and uniqueness of the steady state.

Proposition 1. *Let Λ be a stochastic matrix with positive elements, then:*

1. *There exists $v^\star \in Int(S)$, such that $\Lambda v^\star = v^\star$, that is, $v^\star$ is a steady state.*
2. *The steady state $v^\star$ is unique, that is, if $v, v\prime \in S$ are such that $\Lambda v = v$ and $\Lambda v\prime = v\prime$, then $v = v\prime$.*
3. *For any $v \in S$ $\lim_{m\to\infty} \Lambda^m v = v^\star$, where Λ^m denotes m-power of Λ.*

Proposition 1 is a direct consequence of the Perron-Frobenius theorem; under these[A1] assumptions, it states that 1 is the eigenvalue of Λ, which spans 1-dimensional eigenspace that crosses the interior of S, such that the norm of all other eigenvalues of Λ is strictly less than 1. In the appendix, [A2]we have provided a simple proof of Lemma 1, considering the Brouwer fixed-point theorem.

In Preposition 1, we derived the existence and uniqueness of the steady state for a predefined transition matrix Λ between time s and t, that is, $\Lambda = \Lambda(s,t)$. In this context, we question whether the derived steady state will be the same when considering transitions during a different period of time; say data on flows is available on quarterly basis rather than monthly. Note that if we triple the time interval, the transition matrix becomes Λ^3, that is, $\Lambda \cdot \Lambda \cdot \Lambda$; thus there exists a variation in the flow matrices related to the periods of different lengths.

We consider the process to be stationary if $\Lambda(s,t) = \Lambda(s\prime, t\prime)$ for any $t > s \geq 0$, $t\prime > s\prime \geq 0$, such that $t - s = t\prime - s\prime$. Therefore, transition is only dependent on the length of the time interval during measurement of the flow.

Proposition 2. *Suppose* $\{\Lambda(s,t) : t > s \geq 0\}$ *describes a stationary process, assuming that for any* $s \geq 0$

$$\Lambda(s,t) \to I_n when\ t \to s^+,$$

where I_n *denotes the identity matrix. For any* $s \geq 0$, $t, t\prime > s$, *we define* $v_t^\star$ *and* $v_{t\prime}^\star$ *as the steady states for the matrices* $\Lambda(s,t)$ *and* $\Lambda(s,t\prime)$, *respectively. Hence,* $v_t^\star = v_{t\prime}^\star$.

The proof of Proposition 2 has been provided in the appendix.

Proposition 2 shows that a steady state does not depend on the frequency of data provided that the process is stationary and $\Lambda(s,t) \to I_n$. Although the process is required to be globally stationary in the proposition, we want the derived steady state to be meaningful in an empirical sense, such that the flows change slowly for the economy to approach its steady state over a short period of time. The second condition asserts that the process is continuous; in the case of a stationary process, it is sufficient to assume that it is right-continuous in 0, that is, when $s = 0$. Considering Shimers (2012) model of the economy, this condition is satisfied, as it implies that with the time period converging to 0, nobody changes their current state.

Proposition 2 allows us to calculate the steady state directly from The observed data, relaxing Shimer's assumption that the flow matrices are a result of the Poisson arrival process.

Finally, we relax the assumption regarding the constant size of an economy, and allow for the entry to and exit from the economy. Entries and exits provide cyclic shocks to the economy which can be structurally inconsistent with the current economy state; hence, they can continuously displace the economy from its equilibrium. We are interested in deriving the steady state for such a system, if it exists, and analysing the impact of cyclic shocks on the economy equilibrium.

In the context of the labour market, we can consider a situation where a majority of the new workers are entering the state of employment, while a majority of the workers exiting the market exit from the state of inactivity. In such a case, the new steady state will show higher rates of employment and lower rates of inactivity, as compared to the model with no shocks.

It is assumed that the cyclic shocks related to an entry to and exit from the economy do not change the probabilities of transition between the states for the participants who are already in the market.

As mentioned above, the state of the economy at time t ($t \geq 0$) is formally described by an n-dimensional state vector $v(t)$. Transition between elementary states is described by the system of $n \times n$ flow matrices $\{\Lambda(s,t) : t > s \geq 0\}$, such that $v(t) = \Lambda(s,t)v(s)$ for all $t > s \geq 0$ satisfies the consistency condition. Considering Proposition 2, we can observe the economy from discrete moments of time without a loss of generality. To simplify our considerations, we assume the distance between consecutive moments to be constant. At moments $t = 0, 1, 2, \ldots$, the economy experiences shocks related to entries and exits. Entry to the economy is an n-dimensional vector w_i^{in}, such that for all $i \leq n$, $w_i^{in} \geq 0$ denotes the number of new participants entering each state. Similarly, exit from the economy is an n-dimensional vector w_i^{out}, such that for all $i \leq n$, $v_i \geq w_i^{out} \geq 0$ denotes the number of participants who exit from each state. Let us define the shock as $w = w^{in} - w^{out}$. The evolution of the economy is described by the following equation:

$$v(t) = \Lambda(t-1,t)v(t-1) + w(t)$$

Note that under the above assumptions, if $v(t-1)$ is a state of the economy, then $v(t)$ is also a valid state of the economy belonging to a rescaled simplex S, that is, $S = \{v(t) : \sum_{i=1}^{n} v_i(t) = N(t), v_i(t) \geq 0\}$. Let Λ and w denote the flow matrix $\Lambda(t-1,t)$ and shock vector $w(t)$, respectively. The steady state $v^{\star\star}$ is defined as $\Lambda v^{\star\star} + w = \alpha v^{\star\star}$, where α is a rescaling factor which depends on $N(t-1)$ and w, that is, $\alpha = (N(t-1)+w)/N(t-1)$. The following proposition ensures the existence and uniqueness of the steady state.

Proposition 3. *Let Λ be a stochastic matrix with positive elements and w be a shock vector, then:*

1. *There exists $v^{\star\star} \in int(S)$, such that $\Lambda v^{\star\star} + w = \alpha v^{\star\star}$, that is, $v^{\star\star}$ is a steady state.*
2. *The steady state $v^{\star\star}$ is unique, that is, if v and $v\prime$ are such that $\Lambda v + w = \alpha v$ and $\Lambda v\prime + w = \alpha v\prime$, then $v = v\prime$.*

The proof is analogous to the proof of Proposition 1 and is provided in the appendix.

3. Example

Shimer's (2012) model can be considered as a special case of our generalised model, where $n = 3$ and the distinct states are employment, unemployment, and inactivity. As observed by Shimer (2012), the inflow and outflow of each state in the steady state are equal; hence, the steady state can be derived by solving the following set of linear equations:

$$\begin{aligned}
(\lambda_t^{EU} + \lambda_t^{EI})e^\star &= \lambda_t^{UE}u^\star + \lambda_t^{IE}i^\star \\
(\lambda_t^{UE} + \lambda_t^{UI})u^\star &= \lambda_t^{EU}e^\star + \lambda_t^{IU}i^\star \\
(\lambda_t^{IE} + \lambda_t^{IU})i^\star &= \lambda_t^{EI}e^\star + \lambda_t^{UI}u^\star,
\end{aligned}$$

under the condition: $e^{\star} + u^{\star} + i^{\star} = 1$. The Proposition 3 ensures the existence and uniqueness of solutions to the extended gross flows model that allows for inflows to and outflows from the economy. The analogous system of linear equations to Shimer's (2012) takes the following form:

$$\begin{aligned}
(\alpha - 1)e^{\star\star} &= \Lambda^{UE}u^{\star\star} + \Lambda^{IE}i^{\star\star} + e^{in} - (\Lambda^{EU} + \Lambda^{EI})e^{\star\star} - e^{out} \\
(\alpha - 1)u^{\star\star} &= \Lambda^{EU}e^{\star\star} + \Lambda^{IU}i^{\star\star} + u^{in} - (\Lambda^{UE} + \Lambda^{UI})u^{\star\star} - u^{out} \\
(\alpha - 1)i^{\star\star} &= \Lambda^{EI}e^{\star\star} + \Lambda^{UI}u^{\star\star} + i^{in} - (\Lambda^{IE} + \Lambda^{IU})i^{\star\star} - i^{out}
\end{aligned}$$

under the condition: $e^{\star\star} + u^{\star\star} + i^{\star\star} = N(t-1)$.. [A3]

4. Conclusions

In this study, we suggested three propositions that allowed us to generalise Shimer's (2012) gross flow model in three directions. First, due to the generalisation, our model was not limited to the three states of the economy, as considered by Shimer (2012). Second, we relaxed the assumption of flow dynamics as described by the Poisson arrival process. Third, we allowed the size of the economy to change over time. As an example, we demonstrated the Shimer (2012) model in a more generalised context. Our results are general and not dependent on the labour market conditions. However, our analysis on the labour market serves as a practical example.

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Appendix with Proofs

Proof of Preposition 1:

Set S is convex and compact, $f : v \to \Lambda v$ is continuous and $f(S) \subseteq Int(S)$ hence by the Brouwer Fixed Point Theorem there exists $v^\star \in S$ such that $\Lambda v^\star = f(v^\star) = v^\star$ and $v^\star = f(v^\star) \in Int(S)$. This proves point 1.

Suppose the steady-state is not unique. Let $v, v' \in Int(S)$ be two different vectors such that $\Lambda v = v$ and $\Lambda v' = v'$. Then for any θ, vector $\theta v + (1-\theta)v'$ is also a steady-state. Indeed, $\Lambda(\theta v+(1-\theta)v') = \theta\Lambda v+(1-\theta)\Lambda v' = \theta v+(1-\theta)v'$. Take θ such that $\theta v+(1-\theta)v' \in S - Int(S)$. Then by point 1, vector $\theta v+(1-\theta)v'$ cannot be a fixed point. Contradiction. This proves point 2.

Note that there is $c < 1$ such that for any $v, v' \in S$

$$||\Lambda v - \Lambda v'|| \leq c \sup_{w,w' \in S} ||w - w'||$$

hence by the law of induction $||\Lambda^m v - \Lambda^m v'|| \leq c^m N\sqrt{2}$. Now take $v \in S$. We shall show that $\Lambda^m v \to v^\star$, when $m \to \infty$ Indeed, $||\Lambda^m v - \Lambda^m v^\star|| \leq c^m N\sqrt{2}$ and $\Lambda^m v^\star = v^\star$ hence

$$||\Lambda^m v - v^\star|| \leq c^m N\sqrt{2}.$$

Since $\lim_{m\to\infty} c^m N\sqrt{2} = 0$, then $||\Lambda^m v - v^\star|| \to 0$. This proves point 3. ■

Proof of Preposition 2:

Let $\varepsilon > 0$. We shall show that $||v_t^\star - v_{t\prime}^\star|| < \varepsilon$. Without loss of generality we can assume that $s = 0$. Take $k \in \mathbb{N}$ such that $||\Lambda^m(0,t)v_{t\prime}^\star - v_t^\star|| < \varepsilon/2$ for any $m \geq k$ (1). The existence of such k follows from Proposition 1 point 3, as for any $v \in S$, $\Lambda^m(0,t)v \to v_t^\star$ when $m \to \infty$. Take $\delta > 0$ such that $||\Lambda(0,\tau)v_{t\prime}^\star - v_{t\prime}^\star|| < \varepsilon/2$ for all $\tau < \delta$ (2). This follows from our assumption that $\Lambda(0,\tau) \to \mathbb{I}_n$ when $\tau \to 0^+$.

Consider two cases. Case 1: t/t' is a rational number. In this case, we can take natural numbers n', m' such that $m't - n't' = 0$. Let $m = km'$ and

$n = kn'$. Then $mt - nt' = 0$ and $m \geq k$. Case 2: t/t' is irrational. Then $\{x \in \mathbb{R} : x = mt(mod)t', m \in \mathbb{N}\}$ is a dense subset of the interval $[0, t')$ and we can take natural numbers n and m such that $m \geq k$, $mt > nt'$ and $mt - nt' < \delta$.

In any case, let $\tau = mt - nt'$ and set $\Lambda(0, \tau) = \mathbb{I}_n$, if $\tau = 0$. Note that $\Lambda(0, nt') = \Lambda^n(0, t')$, $\Lambda(0, mt) = \Lambda^m(0, t)$ and $\Lambda(0, mt) = \Lambda(0, \tau)\Lambda(0, nt')$. Then

$$||v_t^\star - v_{t'}^\star|| \leq ||v_t^\star - \Lambda(0, mt)v_{t'}^\star|| + ||\Lambda(0, mt)v_{t'}^\star - v_{t'}^\star||$$

where $||v_t^\star - \Lambda(0, mt)v_{t'}^\star|| = ||v_t^\star - \Lambda^m(0, t)v_{t'}^\star|| < \varepsilon/2$ by (1) and $||\Lambda(0, mt)v_{t'}^\star - v_{t'}^\star|| = ||\Lambda(0, \tau)\Lambda(0, nt')v_{t'}^\star - v_{t'}^\star|| = ||\Lambda(0, \tau)v_{t'}^\star - v_{t'}^\star|| < \varepsilon/2$ by (2) and the assumption that the process is stationary. Thus, $||v_t^\star - v_{t'}^\star|| < \varepsilon$. ■

Proof of Preposition 3:

The proof is analogous to proof of preposition 1. Set S is convex and compact, $f : v \to (1/\alpha)(\Lambda v + w)$ is continuous and $f(S) \subseteq Int(S)$ hence by the Brouwer Fixed Point Theorem there exists $v^{\star\star} \in S$ such that $(1/\alpha)(\Lambda v^{\star\star} + w) = f(v^{\star\star}) = v^{\star\star}$ and $v^{\star\star} = f(v^{\star\star}) \in Int(S)$. This proves point 1.

Suppose the steady-state is not unique. Let $v, v' \in Int(S)$ be two different vectors such that $\Lambda v + w = \alpha v$ and $\Lambda v' + w = \alpha v'$. Then for any θ, vector $\theta v + (1-\theta)v'$ is also a steady-state. Indeed, $\Lambda(\theta v + (1-\theta)v') + w = \theta(\Lambda v + w) + (1-\theta)(\Lambda v' + w) = \theta\alpha v + (1-\theta)\alpha v' = \alpha(\theta v + (1-\theta)v')$.

Take θ such that $\theta v + (1-\theta)v' \in S - Int(S)$. Then by point 1, vector $\theta v + (1-\theta)v'$ cannot be a fixed point. Contradiction. This proves point 2. ■

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