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Enhanced Index Replication Based on Smart Beta and Tail-Risk Asset Allocation

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Abstract: The following research paper's main goal is to create an algorithmically managed ETF, which tracks the SPX index and provides a Smart Beta exposure. Authors apply the following simple index replication methods: partial correlation, non-negative least squares, beta coefficient, and dynamic time warping. First, authors are trying to reverse engineer the Index Tracking process in an automated and fair manner - taking into account e.g. transaction costs. Additionally, authors apply a constraint to the total number of assets used in the replication process, which is limited to the certain N. Then, authors develop a Smart Beta framework based on limiting the negative tail-risk. The positive excess return (alpha) is captured and used to compensate for the underperformance of the replicated Index or paid in a form of a dividend. Moreover, with the enhancement methods applied (Kurtosis/Skewness and Excess Return Cushion (ERC) enhancements), the authors' main goal is to keep the Tracking Error (TE) on a fixed level, although with a significant overweight on the Positive TE and underweight on the Negative TE. In the research paper, the data from 04-Jan-2016 to 31-Dec-2020 is used as the training window, while the first quarter of the year 2021 (Q1 2021) is used as an out-of-sample and out-of-time testing period. Additionally, the authors measure the replicated Index's performance compared to the SPY, VOO, and IVV ETFs. Authors find a piece of empirical evidence that it is possible to track the SPX Index within the limits of 4-5% TE with the limited number of assets. Moreover, after the implementation of alpha accumulation, the authors outperform the benchmark ETFs in terms of minimizing the TE but did not succeed in providing statistically significant returns better than the SPX Index.

Keywords: exchange-traded funds, enhanced index replication methods, smart beta, asset allocation, partial correlation, non-negative least squares, dynamic time warping

JEL codes: C4, C14, C45, C53, C58, G13

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Introduction

The ETF industry has a relatively dynamic history. Recently, investment managers started to apply a more active approach into – what has started as – a passive investing. Although institutional and individual investors have put much value (in dollar terms) into single asset or single index exposure, the tracking approach alone is simply not enough to remain competitive especially in a low interest rates environment with high momentum factor on individual stocks. This paper is trying to focus on a single index, but limit the number of assets used for the replication. Limiting the overall number of assets purchased, should have a positive influence on transaction costs, especially for individual investors or funds with high trading fees. Moreover, to build competitiveness, we develop a methodology that will allow investors to stay within the Tracking Error limits, but overweight the positive deviation in comparison to the negative one. In absolute terms, the error of the replicated index to the SPXTR Index stays the same, but the investor gains and accumulates significant alpha over time in a form of *excess return cushion* (ERC). Such positive excess returns can be used to cover future underperformance or paid out in a form of dividend. To reiterate, the main aim of this paper is twofold:

(1) create a robust replication of the SPXTR Index with a limited number of components, where the TE should be comparable with the most popular ETFs,

(2) successfully apply the enhancement (Smart Beta) methods, in order to decrease the level of negative risk associated with the returns of such replication.

The final simple replication should be comparable to the SPXTR Index as much as possible. The enhanced replication on the other hand, should have a significant overweight on the Positive TE in comparison to the Negative TE, while the Absolute TE stays more or less the same. Based on the above statement, we should be able to construct the Main Hypothesis (H1) of the research in a more technical manner as well as five supporting Research Questions (RQ).

H1: With enhancement methods applied, it is possible to create a synthetic index of N components[1] with absolute Tracking Error similar to benchmark ETFs[2]. Such Index will have a return distribution significantly[3] skewed to the right in comparison to the SPXTR Index, therefore positive deviations

[1] The number N is smaller than the total number of S&P 500 Index constituents in every holding period.

[2] The TE limit is defined as $[0.5 * TE_L; 2.0 * TE_H]$ where TE_L is the lowest, TE_H is the highest TE among benchmark ETFs.

[3] The statistical significance is verified with a two sample $t - test$ for difference of means with 95% confidence interval.

will be more frequent than negative ones.

RQ1: *Without the enhancement methods (Kurtosis, Skewness or ERC), it is possible to create a replicated index using N components that will have a similar return distribution as the SPXTR Index, with statistical significance, applying the statistical tests mentioned in H1.*

RQ2: *Without the enhancement methods (Kurtosis, Skewness or ERC), it is possible to create a replicated index using N components that will have a similar Tracking Error as benchmark ETFs, within the assumed limits mentioned in H1.*

RQ3: *With the enhancement applied (Kurtosis or Skewness), it is possible to create a replicated index using N components that without ERC will have a different return distribution in comparison to the SPXTR Index, with statistical significance, applying the statistical tests mentioned in H1.*

RQ4: *With the enhancement applied (Kurtosis or Skewness), it is possible to create a replicated index using N components that without ERC will have a higher Positive Tracking Error than benchmark ETFs in absolute terms.*

RQ5: *While the above (RQ4) is true, the replicated index with ERC will have the Absolute TE within the allowed limits.*

To achieve that, we use daily prices from Yahoo Finance sourced with the `getSymbols()` function available in the `quantmod` package. In-sample, training data consists of 5 years from 2016-01-04 to 2020-12-31. First quarter of year 2021 is used for out-of-sample, out-of-time testing of the method with best performance. We construct the synthetic index based on Adjusted Close prices for S&P 500 Index constituents and S&P 500 Total Return Index. As benchmark ETFs we use SPY, VOO and IVV which are S&P 500 Index trackers from SPDR, Vanguard and BlackRock (iShares), respectively, as these indices represent a sample of the most popular ETFs from the biggest institutions.

To replicate the SPXTR Index we allocate assets in the portfolio using Partial Correlation, Non-Negative Least Squares and Beta as well as Dynamic Time Warping. For the first two methods, we use the value of correlation coefficient or regression coefficient, to adjust the weights in the replicated index portfolio. We apply the biggest weights to assets with the biggest correlation or regression coefficient values. Similarly we approach the Beta method. However instead of applying the highest coefficient to highest Beta, we transform the coefficient, to apply the biggest weights on the assets with Beta (measured against the SPXTR Index) close to one. Last, for Dynamic Time Warping we look at the smallest distance value, which is the output of the applied algorithm. In other words, we reward assets with the smallest distance to the SPXTR Index. For the enhancement, we will adjust the initial weights based on the analysis of Skewness and Kurtosis for individual assets in comparison to the same moments for SPXTR Index.

For performance metrics, we use Annualized Return (ART), Cumulative Return (CRT), Annualized Standard Deviation (ASD), Annualized Sharpe Ratio with RFR (ASR), Maximum Drawdown (MDD), Annualized Information Ratio (AIR), Absolute Tracking Error (aTE), Positive Tracking Error (pTE) and Negative Tracking Error (nTE). Metrics are calculated with the help of `PerformanceAnalytics` package available in R. For weight allocation we additionally use `ppcor 1.1.0`, `nnls 1.4.0` and `dtw 1.2.3` packages. Raw data is sourced and manipulated with the tools available in `tidyverse 1.3.1` and `tidyquant 1.0.3`. The output was generated using `R version 4.0.5` and `rmarkdown 2.8.0`.

This research paper is constructed as follows. First section presents work related to ETFs, methods used to build the replicated index, Smart Beta approach and Tail Risk filtering methods. Next, we present the dataset thoroughly and comment on all transformations that were made to raw data. Third section presents the methodology, where all variables, which may spark the reader's interest are presented and defined. Section fourth and fifth present results on simple index replication and enhancement methods, respectively. Next, final results on testing data are presented. Second to last section presents a thorough sensitivity analysis, and the last section concludes.

1 Literature Overview

In this Chapter we present the literature mainly connected to the topic of ETFs and an application of Smart Beta enhancements. Additionally we show the inspiration behind applying selected methods such as Partial Correlation or Dynamic Time Warping for asset allocation purposes in our proposed portfolios. We are aware of the fact that the Smart Beta methods are thoroughly examined through the scientific papers, but we believe that based on the related work, there is still a space for improvement, especially when we combine the methods explored in standard ETF replication literature with novel Smart Beta approaches and other quantitative methods to optimally allocate the portfolio.

From the perspective of index replication process, there is a substantial research done on traditional and synthetic replication methods. Fundamentally, there are two ways to approach a traditional replication. First method – full replication – assumes that the portfolio should consist of all assets that are present in a given benchmark. In contrary to that, another approach that we present in this paper is sample replication, where the portfolio of replicated index consists only of a fraction of all assets. Regardless of investing in full or a fraction of assets, traditional methods assume, that the investment is done on physical assets, that is why we usually call this method a physical replication. However, as mentioned at the beginning of this paragraph, there is also a synthetic way to replicate the benchmark index. The returns or performance is – in short – delivered in a form of *performance swap* (Lyxor 2019), but this is not the goal of this research. As introduced, we are interested in sample physical (traditional) replication method.

Less from the theoretical perspective and more from the empirical one, Naumenko and Chystiakova (2015) shows that the synthetic ETFs have higher tracking errors than their physical counterparts. In line with these findings is the research done by Fassas (2014), where he proves that the synthetic (swap-based) ETFs are underperforming the traditional ones in tracking behaviour, measured by tracking error. The results were found to be statistically significant.

Regardless of physical or synthetic replication, another research (Frino and Gallagher 2001), shows and highlights the problems of the tracking as a whole. The author argues, that although the theory is simple and understandable, there is a significant difference between paper portfolio used for tracking and the real implementation, where the tracking error is inevitable due to market frictions and costs connected to achieve substantial results. Nevertheless, research provides an evidence that the funds tracking the S&P 500 Index outperformed the active-managed funds after expenses, for the period analyzed.

Simple Exchange-Traded Funds (ETFs), have a relatively long history on capital markets. Their emergence is dated to around 1990s, where initially they were supposed to be a solution for individual investors, providing a passive return from a single index exposure, e.g. the SPXTR Index. However, soon the passive and index-tracking behaviour was simply not enough and various enhancements were created. Nowadays, there are multiple ETFs that are e.g. tracking a thematic indices, where the benchmarks are from a certain niche industry such as L&G Battery Value-Chain UCITS ETF. On the other hand, there are various enhancements applied to a simple ETFs, that can track e.g. S&P 500 Index, but are trying to provide additional positive excess return. Smart Beta can be considered such a development, where managers are seeking more active approach, more similar to mutual funds.

According to various definitions, Smart Beta can be defined as *enhanced indexing strategy that* *seeks to exploit certain performance factors in an attempt to outperform a benchmark index.* Many researchers have been intrigued by this and attempted to explore whether this solution is sustainable for the longer term or not. Malkiel (2014) asked the question "Is Smart Beta Really Smart?" and he found out that such portfolios cannot consistently outperform their benchmark. Similarly, Glushkov (2015) examined a sample of over 150 equity Smart Beta ETFs. In his research, he explores the behaviour of such instruments from 2003 to 2014, and finds no empirical evidence that Smart Beta can outperform the benchmark on a risk-adjusted basis. More recent study conducted by Mateus, et al. (2020), after an analysis of a similar number of US ETFs finds that just 40% of funds are outperforming their traditional counterparts after expenses. Interestingly, he finds that there is a substantial persistence of underperformance connected to Smart Beta. These researchers have presented a meta-analysis of the Smart Beta performance, but there is a substantial literature on the proposed Smart Beta methodologies, that are considered to be innovative.

Group of researchers (Fons et al. 2021) argue, that Smart Beta strategies are good long term performers, but suffer in the short term. They aimed to address this problem by implementing a dynamic asset allocation system based on the Hidden Markov Models (HMM). Authors provided an evidence of an improvement in risk-adjusted returns especially on more aggresive portfolios. Additionally, they proposed an innovative Smart Beta asset allocation framework based on Feature Saliency HMM (FSHMM) algorithm, that provides an additional feature selection stage during the HMM training. This approach again provided an uplift on the performance.

Although we see that there is an empirical evidence of Smart Beta ETFs consistently underperforming traditional solutions - simple and passive index-tracking ETFs, there are proofs of novel Smart Beta approaches, that can outperform their benchmarks. That being said, the majority of studies showing a consistent underperformance should not discourage us, as in this research paper, we intend to first reverse-engineer the ETF creation process with a highly limited number of assets - that is, create our own methodology and framework from scratch, aiming to provide results similar to benchmark ETFs - and only then, try to generate additional alpha, similarly to what Smart Beta is trying to achieve. However, we intend to accumulate this positive excess return, to cover for underperformance and - optionally - paid out in a form of a dividend. We are not assuming the systematic existence of overperformance, but we treat it as an advantage of our approach, that is not granted.

As we aim to generate and provide excess positive return, it is worth looking at the research connected to the methods that we are interested in applying. Among the quantitative finance and financial market researchers, there is a high amount of papers dedicated to various algorithms and numerical methods applied to time series and to portfolio optimization. One example can be a method of Partial Correlation applied to portfolio theory (Nadler and Schmidt 2016). Researchers find that the asset allocation based on Partial Correlation can outperform its Pearson Correlation counterpart. Moreover, there is a strong evidence that financial markets tend to create patterns (Coelho 2012). Such patterns can be exploited if approached reasonably. The special attention is granted towards Dynamic Time Warping which seems to have significant results, when applied to time series from financial markets. Additionally, computer science researchers (Nicolae et al. 2016) use Dynamic Time Warping for classifiaction purposes as well. Authors find an evidence, that the method yields efficient classifiers. Other than that, there is an example of a machine learning approach applied to ETFs in (Baek et al. 2020), where the researchers are using the state-of-the-art methods for predicting the market movements.

Based on the work related to the topic, we strongly believe that the experiment that we aim to do can provide strong empirical evidence in favour of applying selected methods to the problem stated in the research. In the literature, there is a strong evidence against the Smart Beta approach. Especially visible is the fact, that Smart Beta funds, tend to underperform their simple counterparts. Nevertheless, what we aim here, is not to create another Smart Beta approach, but rather to re-invent the framework. We believe, that our approach is based heavily on the disadvantages of the current framework, and can provide a valuable advancement into these types of assets. Moreover, we find an evidence of various methods applied to financial markets data with a significant success, especially Partial Correlation and Dynamic Time Warping.

2 Data Description

We use publicly available data throughout the whole research paper. Index that will be replicated is S&P 500 Total Return (SPXTR Index), available on Yahoo Finance under the `^SP500TR` ticker. To replicate the Index, we source Adjusted Close data for each of the 500 constituents from 2016-01-04 to 2020-12-31. To download the data efficiently we use the `getSymbols()` function from the `quantmod` package available in R. To measure the relative performance versus popular benchmarks, we additionally source SPY, VOO and IVV Adjusted Close price data, using the aforementioned methods.

Downloaded price information is available for the whole period for which the security was listed on a public exchange. As we are interested in the SPXTR Index replication, we add an additional filter in the form of NA imputation, for the periods where certain equity was listed, but were not considered a part of the S&P 500 Index. Tesla ($TSLA) can be such an example. We do not adjust the dataset to control for the survivorship bias, as we cover the last 5 years of Index history, where no significant delisting or bankruptcy events have taken place among the most liquid companies that we are interested in. However, we are aware of the fact that such an event could potentially affect the results.

To filter only the N constituents, we use trading volumes to build the ranking. Volume is sourced together with Adjusted Close data for each of the constituents. It is expressed as the number of shares circulating on a given day, so to express this in dollar terms, we multiply the volume on each day by the Close price on that day, we deliberately not multiply by Adjusted Close. We consider the biggest trading volume as the first rank, second biggest as the second rank, etc.

For the whole 5 year period we have 1259 observations (trading days) and 504 securities. As mentioned above, we treat the period from 2016-01-04 to 2020-12-31 as a training sample, and the first quarter of the year 2021 (2021-01-04 - 2021-04-01) as a testing sample.

3 Methodology

3.1 General Methodology

In this section we shed more light on the assumptions connected to training data experiments with various replication methods. We start with the N number of securities taken into account. To set this variable on a fixed level we choose $N = 25$ as our baseline approach. From the perspective of individual investors, number 25 seems like a viable quantity of securities to purchase. Next, the rebalancing window. We aim to rebalance our portfolio on a fixed-term basis, we initially set this parameter to 126 trading days, which should account for approximately a half of the trading year. From the operational and transaction costs perspective semi-annual rebalancing seems fair as well. Given that the number of rebalancing periods for the training sample is greater than 0, we test each method repeatedly. Under certain algorithms, we test the same assumptions (parameters) for different time windows, as we rebalance and recalculate the weights in the replicated index.

For the transaction costs (TC), we apply a linear approach and set the unit TC to 1%. That said, TC are not increasing or decreasing due to higher or lower volume of trading, but are growing and decreasing linearly according to portfolio turnover. We assume that changing 1% of a portfolio costs us 0.01% of total assets, regardless of buying or selling. That being said, if we have to rebalance the whole portfolio, that is first sell 100% and later buy 100% of the portfolio, it will cost us 2% of total assets. While calculating the rebalancing costs, we take into account the security price growth or decrease over time. In other words, we apply TC after calculating the intrinsic change of security allocation (as % of portfolio) during holding time, due to its price increase or decrease for that period, and then calculate the amount of allocation we need to sell or buy. Now, let us describe the process of portfolio weight fitting.

(1) We start by setting a rebalancing window (expressed in trading days) and a maximum number of assets used to replicate the index. This amount of assets is fixed to the whole period – it does not change between different rebalancing periods. To remind the reader, at this point we have prices and returns only for the assets, that we are able to invest in for each period, meaning the data consists of the assets that are listed and considered a part of the S&P 500 Index only.

(2) We calculate the average trading volume in dollar terms. Its rolling (moving) average window is based on the rebalancing period length. For the ranking process, we select the moving average value on the rebalancing day date and create a ranking from 1ˆst to 25ˆth stock in terms of average amount traded (proxy for liquidity). These stocks are given the *available to trade* flag for the selected rebalancing period, while the rest are restricted.

(3) We proceed to the process of setting the weights, which is significantly different for each of the methods described in the next section.

After this process, we recalculate for the trading costs, which are applied to daily returns on the day of rebalancing. Next, we proceed to the evaluation based on a predefined set of metrics, that are explained below. All metrics are based on the `PerformanceAnalytics` package available in R. MDD is based on Ryś and Ślepaczuk (2019). For metrics listed below and algorithms that we work with, we need to create daily returns data for every constituent, the SPXTR Index and three ETFs as well as cumulative return and other performance metrics, to achieve that, we define:

$$N \text{ is the number of observations, where each Day is separate observation}$$

$$T \text{ is the number of trading days within a trading year, which is set to } 252$$

RFR_t *is daily Risk Free Rate, which is set to* 1.5% *per annum*

P_t *is the Daily Adjusted Close Price for day* t

R_t *is the Daily Return for day* t *expressed as* $P_t/P_{t-1} - 1$

R_a, R_b *are vectors of Daily Returns for the investment and benchmark, respectively*

ART – *annualised rate of return*

CRT – *cumulative rate of return*

ASD – *annualised standard deviation*

ASR – *annualised Sharpe ratio*

MDD – *maximum drawdown*

AIR – *annualised information ratio*

$$ART = prod(1 + R_a)^{\frac{T}{N}} - 1 \tag{1}$$

$$CRT = prod(1 + R_a) - 1 \tag{2}$$

$$ASD = \sqrt{Var(R_a)} * \sqrt{T} \tag{3}$$

$$ASR = \frac{prod(1 + R_t - RFR_t)^{\frac{T}{N}} - 1}{\sqrt{Var(R_a)} * \sqrt{T}} \tag{4}$$

$$MDD = sup_{x,y \in \{[t_1,\, t_2] \,:\, x \le y\}} \frac{P_x - P_y}{P_x} \tag{5}$$

$$AIR = \frac{ART}{ASD} \tag{6}$$

$$aTE = \sqrt{\sum \frac{(R_a - R_b)^2}{N\sqrt{T}}} \tag{7}$$

$$pTE = \sqrt{\sum \frac{(R_a - R_b)^2}{N\sqrt{T}}}\ ,\ only\ for\ observations\ where\ R_a\ >\ R_b \tag{8}$$

$$nTE = \sqrt{\sum \frac{(R_a - R_b)^2}{N\sqrt{T}}}\ ,\ only\ for\ observations\ where\ R_a\ \leq\ R_b \tag{9}$$

We conduct an additional sensitivity analysis at the end of the paper to test various approaches to rebalancing/holding period as well as maximum number of securities used for replication.

3.2 Methodology for Simple Index Replication

The goal of Simple Index Replication is to get as close to the SPXTR Index as possible. As mentioned in Research Questions (1) and (2), we measure the similarity between replicated index and the SPXTR Index by looking at the return distribution and the Tracking Error comparison between replicated index and benchmark ETFs. After creating the equities ranking by Volume, we apply four different methods to pick the most comparable assets out of 500 constituents. We apply these methods to the return vector of each asset. After receiving the output, we standardize weights to sum up to 100%. It is worth noting that we construct Long Only Portfolios, with one exception, that we elaborate on later. Next, we compare return distributions of the SPXTR Index and the replicated index, as well as apply the TC. At the end, we summarize each method in a form of Table with all performance metrics defined above.

3.2.1 Partial Correlation

First, we use Partial Correlation (PCOR) as a weight-creation system. It is important to note that we calculate the PCOR Matrix for the whole sample of assets (that is 500 at each rebalancing day), as the Partial Correlation takes into account not only the correlation between two vectors, but how other variables correlate with them as well. If the investor has a predefined list of maximum assets that are available for investment – that is 25 most liquid stocks in our baseline scenario – he or she needs to know how much certain assets correlate with the SPXTR Index taking into account the broader market perspective as well. In our opinion, the correlation is a straightforward solution to tracking and thus replicating as well. Although the method is not considering the time variable, we believe that it is important to test such a baseline and simple approach as well. Additionally, given the fact that we operate on the rebalancing periods, the time factor is somehow included as well – as we repeat the process of correlation coefficient computation several times in the whole training period.

For each rebalancing period, we look at the data for a certain number of days (126 in baseline scenario) in the past. We calculate the Partial Correlation Matrix, replace negative correlation coefficients with zero and standardize the weights, which are based on correlation coefficients. Therefore, we obtain the standardized weights, that sum up to 100%, for the universe of assets, that positively correlate with SPXTR Index. In other words, if the asset correlates with the SPXTR Index more, it will receive a higher weight.

Although we mentioned that we replace negative correlation with zero, we decided to perform an additional test with Long-Short Portfolio, to test the cost and the impact of skipping the negatively correlated assets in Long-Only Portfolio. The methodology remains the same – if a certain asset has higher correlation (this time in absolute terms) it receives higher weight (positive or negative, where negative means that we are shorting the asset throughout the next holding period). To make things simpler, we apply the same transaction costs approach to shorting. We believe that this is an important test to be done, due to two factors. First, with positive coefficients, we skip negatively correlated assets, which can be used to balance the returns and therefore bring our portfolio closer to the SPXTR Index, which is a desired outcome in the replication process. Second, given that asset has to meet Volume criterion as well as positive coefficient criterion, the number of assets that is in the portfolio during holding periods, often is – as we calculated ex-post – much lower than the assumed 25. For the Partial Correlation Matrix calculation we use the `pcor()` function from the `ppcor` package available in R.

3.2.2 Non-Negative Least Squares

Next, we apply the Non-Negative Least Squares (NNLS) method to find the weights and perform the asset allocation in our replicated index. Although the method is fairly similar to the Partial Correlation in theory, it returns only positive coefficients, which allows us to consider Long-Only Portfolio without sacrificing negative coefficients from the output. In comparison to the PCOR Index, where for Long-Only option we are usually investing in a smaller number of assets than 25, as the number of the most liquid assets with positive correlation coefficient is smaller than our assumed 25, here the number is considerably higher. Nevertheless, there is no certainty that we will have all 25 assets allocated in the portfolio, as the NNLS coefficient can be zero, which would imply the weight that equals zero.

The Non-Negative Least Squares seems to be an appropriate method for the task that we aim to accomplish due to two reasons. First, this is a well-known and reliable econometric method, similar to the Ordinary Least Squares with a high degree of explainability in comparison to state-of-the-art machine learning and deep learning methods. Second, it sets the coefficient proportionally to the explained variance in the dependent variable. Given that, with a sufficient number of assets considered, we can build a portfolio with satisfactory tracking abilities. On the contrary, the method has certain disadvantages that arise with the limit of assets we impose. If the number of stocks available to invest is too narrow, the method may lose on its explainability power, due to the artificial restriction. In theory, the method would perform the best on the full sample of available assets.

Process of weight creation based on the coefficient value as well as the evaluation is the same as described for the previous method. We start by calculating the coefficient vector, then select weights for only the most liquid assets and standardize the weights, such that the sum of weights is 100%. To compute the vector of NNLS coefficients we use the `nnls()` function from the `nnls` package available in R.

3.2.3 Beta Coefficient

After NNLS, we apply the Beta Coefficient (BETA) approach. Although the process of performance evaluation is standardized for each method, here the weights creation algorithm is significantly different from the previous two methods.

First, we calculate the Beta Coefficient between each of the constituents and the SPXTR Index. Next, we invert the coefficient value (divide 1 by the coefficient) to reward assets with Beta coefficient close to one or below, and penalize those which have the Beta coefficient significantly higher than one. Therefore we avoid investing in assets that are much more volatile than the SPXTR Index. Theoretically, obtaining the Beta coefficient that is exactly one, has a very low probability of occuring, but even if that happens, it does not break the weights standardization process and portfolio allocation processes.

This approach is different from the previous, as it applies a more financial approach, rather than econometric. Beta Coefficient is one of the most popular metrics that professional investors look at and it separates the assets more volatile from the ones that tend to move in line with e.g. the benchmark. This method is always in the market, as each asset analysed will have a weight different than zero, with higher weights are set to assets less volatile than the SPXTR Index. To calculate Beta coefficients we use the `BetaCoVariance()` function from the `PerformanceAnalytics` package available in R.

3.2.3 Dynamic Time Warping

Last, we apply the Dynamic Time Warping (DTWA) method. This approach, similarly to the BETA, is always in the market, meaning that the weight output is always higher than zero. It means that we always maintain 25 assets in the portfolio. Similarly as with the Beta Coefficient, we invert the distance from the algorithm output (divide 1 by the distance), to overweight and reward assets with smaller distance in comparison to those which are more distant to the SPXTR Index. This method is the most promising, as it takes into account the sequential character of the time series we analyze. The only way we took into consideration the time factor and the changing relationship between constituents and SPXTR Index in time was through the rebalancing window. Now, with the Dynamic Time Warping method, we are looking into the time factor within the rebalancing window itself. The algorithm is preserving the sequence of observations – returns in this sense – and draws conclusions upon that. This method – as shown in the section dedicated to the work related to the topic – is especially successful in time series pattern recognition, therefore the effect of it will be interesting to examine in comparison to regular financial and econometric approaches. To compute the distances we use the `dtw()` function from the `dtw` package available in R.

In the next chapter, we present the results of methods described above, along with performance measurements, Tracking Error and statistical tests included. Before that, in the next section, we define methods used for the enhancement (Smart Beta), which we apply to the best method for Simple Index replication.

3.3 Methodology for Enhanced Index Replication

During Simple Index Replication, we aimed to get as close to the SPXTR Index equity line as possible. To accomplish this, we tried various methods that look for vector or time series similarities.

First three methods were looking just at the vector of values, without taking into account the sequential character of the data. The fourth method – Dynamic Time Warping – is applying an approach, which seems to be better suited to time series. Now, we present the methodology used for the enhancement of the replication. By applying the enhancement, we aim to remain as close to the SPX as before, but in absolute terms. While the Absolute Tracking Error (aTE) should remain fairly constant, we want to have more positive excess return than the negative. In that sense after enhancement, we will be more likely to see a higher Positive Tracking Error (pTE), and lower Negative Tracking Error (nTE).

We would like to keep aside the pTE in a form of accumulated positive excess return, to compensate for future negative excess return (underperformance). In other words, we are implementing an *excess return cushion* (ERC), that will be more substantial, the higher the accumulated positive excess return (and pTE). To put it in more technical terms, after each day, when our replicated index is outperforming the SPXTR Index, we match the gains and set the positive excess daily return aside. When our replicated index is underperforming, we use the accumulated return to compensate for the loss. If the *cushion* is not enough to compensate, we use it all. During the rebalancing periods, we test two solutions. First, we just rebalance the portfolio and continue. In the second approach, after rebalancing and paying the TC, we pay out the remaining *cushion* that was accumulated during the last holding period in a form of dividend, that is one-off additional positive return during the rebalancing day. To sum up, when we apply the *excess return cushion*, we do not let the synthetic index outperform the SPXTR Index. All alpha is set aside for the future loss compensation or loss compensation and dividend payout during rebalancing days. By applying the enhancement methods and capturing more alpha, we should be able to generate substantial excess positive returns and thus have a bigger amount to compensate for the underperformance. In a perfect solution, with far more positive days than negative (in a sense of excess return), we should always have a sufficient amount of return to compensate for underperforming daily returns. This brings us closer to the desired Tracking Error (as we do not outperform and almost always match the underperformance), and gives investors an additional alpha, which is paid out only during the rebalancing, therefore does not impact the TE; whereas during the holding periods, the replicated index is mimicking the SPXTR Index as good as possible.

That said, after selecting the best Simple Index Replication method, we firstly investigate the effect of *excess return cushion* on Simple Index, and then verify, whether or not our enhancement methods are generating more alpha, therefore keeping aTE unchanged and overweight the pTE in comparison to nTE. Then, we incorporate the *cushion* on Enhanced Index to see what is the final outcome.

For the enhancement, we use the information about the individual asset's kurtosis and skewness in comparison to the SPXTR Index. We use this information as a weight multiplier. In other words, if we find that either kurtosis or skewness values are supportive according to our methodology, we multiply the original weight by a number higher than 1, or lower than 1 otherwise (if we see that the information about kurtosis or skewness is unsupportive according to our methodology). In approach where we use Kurtosis, we look for assets, that have higher Kurtosis in absolute terms, than the SPXTR Index. In other words, we reward assets with more centralized return distribution, or with longer and wider tails. However, if we take into account the fact that usually the right tail is further apart, than the left (or that the average and median return is usually positive for financial assets) it may have the positive influence on the performance. For Skewness, we look for assets with higher positive skew than SPXTR Index has. In that sense, we reward assets, that on average, have more right tail returns, which should positively influence the enhancement and alpha generation.

First, we create the ratio between the asset's kurtosis and the kurtosis of the SPXTR Index. We use that ratio as the multiplier for original weight. The assumption is that if the asset has a higher kurtosis than the SPXTR Index, it will positively influence the tracking mechanism, as we expect more centralised returns and less long-tail negative returns. We use the `kurtosis()` function from the `PerformanceAnalytics` package available in R to compute the kurtosis for each constituent and the SPXTR Index.

Secondly, we calculate the difference between the asset's skewness and the SPXTR Index's skewness. We replace negative results with 0 and add 1 to this value, to reward assets with more positive skew in comparison to the SPXTR Index and to leave other weights unchanged. This method should reward more positively skewed assets, therefore increase the probability of outperforming returns. We use the `skewness()` function from the `PerformanceAnalytics` package available in R.

4 Results of Simple Index Replication

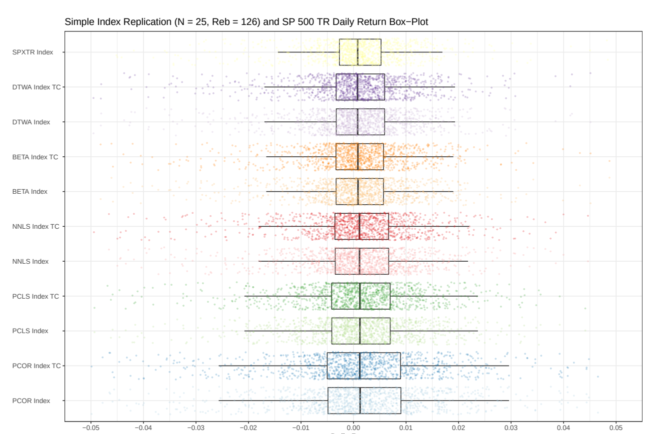
In this section, we present the results of Simple Index Replication, together with the verification of Research Questions (1) and (2) for each method presented. To remind, hypotheses were that:

RQ1: *Without the enhancement methods (Kurtosis, Skewness or ERC), it is possible to create a replicated index using N components that will have a similar return distribution as the SPXTR Index, with statistical significance, applying the statistical tests mentioned in H1.*

RQ2: *Without the enhancement methods (Kurtosis, Skewness or ERC), it is possible to create a replicated index using N components that will have a similar Tracking Error as benchmark ETFs, within the assumed limits mentioned in H1.*

As previously mentioned, we start with 126 trading days rebalancing period and 25 assets in the portfolio. First 126 days of data are used to compute first weights (this period is naturally skipped in results presentation and visualisation. We present each method with and without Transaction Costs to measure the impact of TC on the performance. At the end of this section, we select the best method, and proceed to the enhancement process and the application of *excess return cushion*. As benchmark in this section we use the SPXTR Index and three selected S&P 500 Index ETFs, namely SPY, VOO and IVV. Before we proceed to the detailed evaluation of each metric, below is the Box-Plot with daily returns for each method tested as well as the SPXTR Index, where we see the gradual path towards more comparable daily return distribution for the methods presented in the next sections.

Figure 1. Return Distribution Box-Plot for Simple Indices and the SPXTR Index



Note: Each object on the Figure consists of daily returns for the whole training period. The vertical position inside the Box-Plot is chosen randomly, while the horizontal position is strictly connected to the value of daily return (X-Axis). Returns are truncated to [-0.05, +0.05]. The PCOR, PCLS, NNLS, BETA and DTWA stands for Partial Correlation Index, Partial Correlation Long-Short Index, Non-Negative Least Squares Index, Beta Coefficient Index and Dynamic Time Warping Index, respectively. The TC stands for Index with Transaction Costs included, otherwise the box-plot is for replicated index without TC accounted. SPXTR Index represents data for SP 500 Total Return Index.

4.1 Index based on Partial Correlation (PCOR)

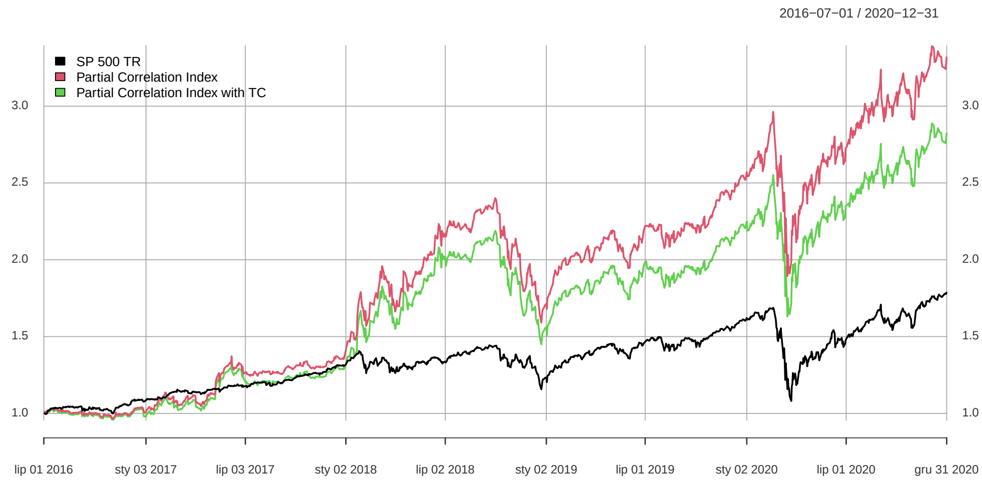
Table 1 proves that the PCOR Index has a strong Annualized Return (ART), and even the risk-adjusted returns (in the form of Annualized Sharpe Ratio, ASR) are better than the SPXTR Index and the benchmarks. From the visual inspection of the Partial Correlation Index fitting, in Figure 2, we see that due to periods of strong outperformance, we failed to remain close and minimize the Tracking Error to the SPXTR Index regardless of introducing the TC. This fact is especially visible on the Figure 5, where we present the rolling correlation between the two time series. We notice that for some periods, the correlation is even negative. Although from the perspective of other strategies, it might be considered a positive scenario, in the replication exercise it certainly is not, here we are optimizing for the TE, which is by far too high from the benchmark ETFs. If the problem lies with the low asset exposure – as mentioned in Methodology, we are not investing in 25 assets for every period – the Long-Short Partial Correlation Index should have lower TE and be closer to the SPXTR Index. The kurtosis of the replicated index seen in Figure 3 is noticeably lower than the SPXTR Index, and even the inclusion of Transaction Costs does not affect the excess positive return significantly enough to bring the TE into desired levels.

Table 1. Performance Metrics for PCOR Index

	Partial Correlation			Benchmark ETFs		
Name	PCOR	PCOR_TC	SPX	SPY	VOO	IVV
ART	0.3058	0.2598	0.1342	0.1336	0.1337	0.1323
CRT	2.3221	1.8270	0.8759	0.8708	0.8717	0.8602
ASD	0.2813	0.2825	0.1932	0.1895	0.1930	0.1933
ASR	1.0180	0.8533	0.6070	0.6157	0.6051	0.5973
MDD	0.3608	0.3608	0.3597	0.3575	0.3620	0.3612
AIR	1.0870	0.9197	0.6944	0.7047	0.6925	0.6845
aTE	0.1670	0.1686	0.0000	0.0116	0.0088	0.0097
pTE	0.1348	0.1340	0.0000	0.0108	0.0073	0.0076
nTE	0.1117	0.1151	0.0000	0.0091	0.0069	0.0081

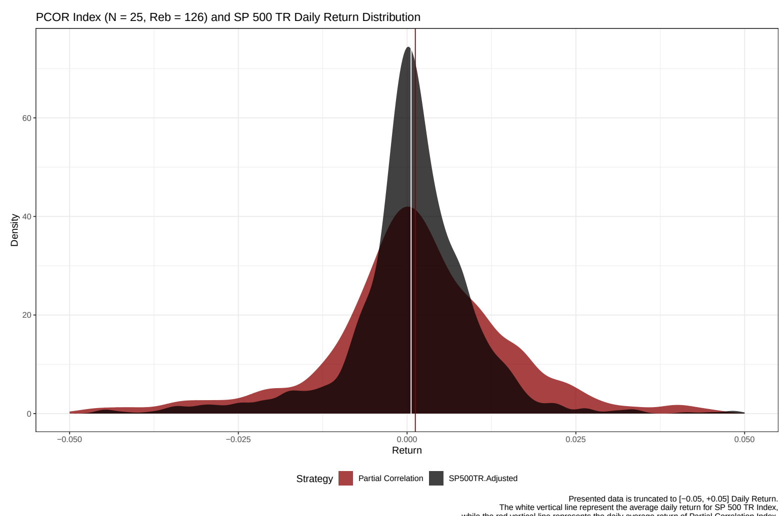
Note: The above table presents the performance metrics for replicated index based on Partial Correlation (Long-Only Portfolio) with and without Transaction Costs included in comparison to the SPXTR Index and three benchmark ETFs. Portfolio allocation is based on the coefficient value, where it rewards higher coefficients. Weights are scaled to sum up to 100

Figure 2. Cumulative Return of the SPXTR Index and PCOR Index with and without TC



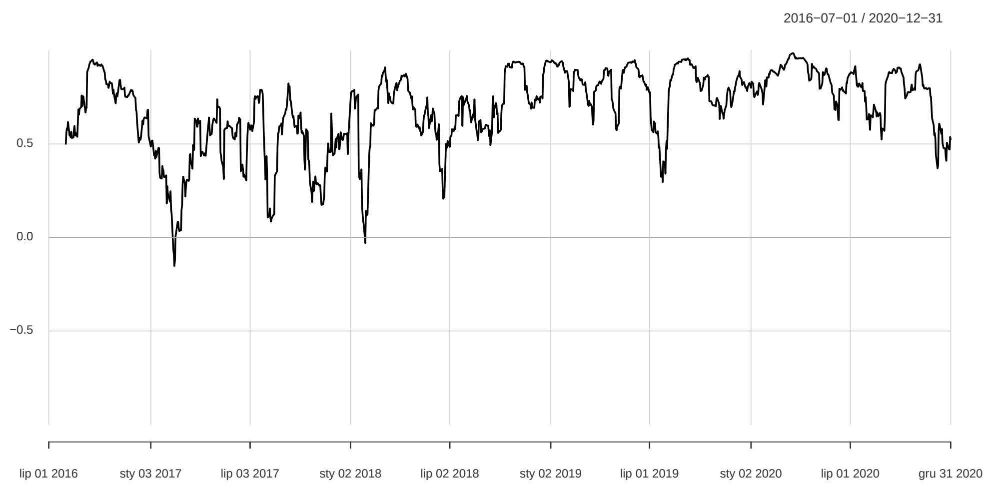
Note: Figure presents the cumulative return for the Index without Transaction Costs, the Index with Transaction Costs and the SPXTR Index. Equity line is calculated by taking the cumulative product of daily returns.

Figure 3. Return Distibutions of the SPXTR Index and PCOR Index



Note: Figure presents the distribution of daily returns for the replicated index with Transaction Costs and the SPXTR Index. Presented data is truncated to [-0.05, +0.05] daily returns. The white vertical line represents the average daily return for the SPXTR Index, while the red vertical line represents the daily average return of the replicated index with Transaction Costs.

Figure 4. Rolling Correlation (n = 21) between the SPXTR Index and PCOR Index with TC



Note: Figure presents the coefficient for rolling Pearson Correlation between the vector of the replicated index daily returns and the vector of the SPXTR Index daily returns. The rolling window for the calculation is set to 21 trading days.

4.2 Index based on Long/Short Partial Correlation (PCLS)

With the inclusion of shorting in the asset allocation of replicated index portfolio, we indeed see that the tracking behaviour is more precise, at least from the visual perspective of analyzing the Figure 5 below. Nevertheless, there is a significant overperformance at the beginning of the training period, which drives the overall cumulative return higher than expected (Table 2). On the Figure 6, the return distribution plot, we hardly notice any difference from the Long-Only Partial Correlation Index. Again, we see the fat tails and lower kurtosis. What is interesting to see is the fact that higher transaction costs associated with potentially higher portfolio turnover (due to enabling short option) is not affecting the drop in ART that much, when we compare replicated index with and without TC. The inclusion of TC brings the replicated index even closer to the SPXTR Index, but not close enough to mimic the SPXTR Index for the period of 5 years. The rolling correlation as seen on the Figure 7 is not negative for any of the rolling calculations, but is still not high enough, as it has low values of approximately 10%, which is not considered a relationship high enough. Although we have managed to lower the Tracking Error by including a shorting option in portfolio allocation, we still have a lot of space for improvement. Let us proceed to the NNLS method.

Table 2. Performance Metrics for PCLS Index

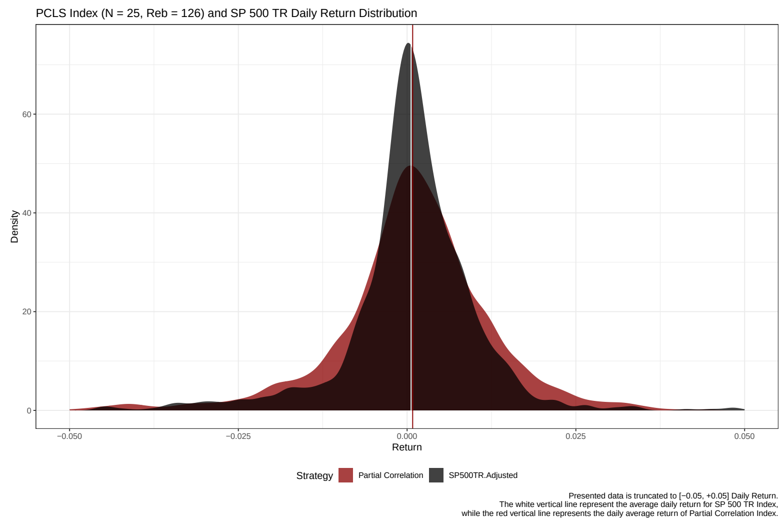
	Partial Correlation			Benchmark ETFs		
Name	PCLS	PCLS_TC	SPX	SPY	VOO	IVV
ART	0.1992	0.1519	0.1342	0.1336	0.1337	0.1323
CRT	1.2649	0.8892	0.8759	0.8708	0.8717	0.8602
ASD	0.2286	0.2316	0.1932	0.1895	0.1930	0.1933
ASR	0.7935	0.5817	0.6070	0.6157	0.6051	0.5973
MDD	0.3916	0.3916	0.3597	0.3575	0.3620	0.3612
AIR	0.8716	0.6557	0.6944	0.7047	0.6925	0.6845
aTE	0.0925	0.0992	0.0000	0.0116	0.0088	0.0097
pTE	0.0709	0.0710	0.0000	0.0108	0.0073	0.0076
nTE	0.0591	0.0740	0.0000	0.0091	0.0069	0.0081

Note: The above table presents the performance metrics for Index based on Partial Correlation (Long-Short Porftolio) with and without Transaction Costs included in comparison to the SPXTR Index and three benchmark ETFs. Portfolio allocation is based on the coefficient value, where it rewards higher coefficients. Weights are scaled to sum up to 100

Figure 5. Cumulative Return of the SPXTR Index and PCLS Index with and without TC

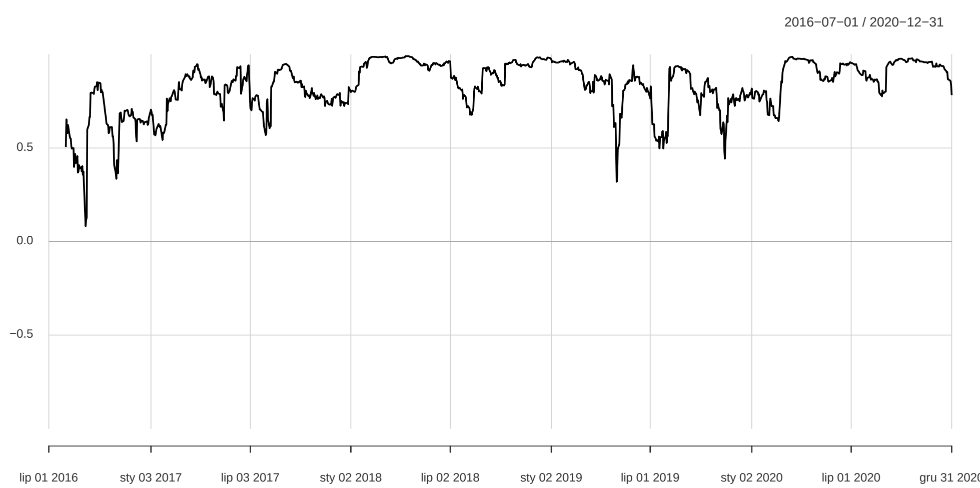
Note: Figure presents the cumulative return for the Index without Transaction Costs, the Index with Transaction Costs and the SPXTR Index. Equity line is calculated by taking the cumulative product of daily returns.

Figure 6. Return Distibutions of the SPXTR Index and PCLS Index



Note: Figure presents the distribution of daily returns for the replicated index with Transaction Costs and the SPXTR Index. Presented data is truncated to [-0.05, +0.05] daily returns. The white vertical line represents the average daily return for the SPXTR Index, while the red vertical line represents the daily average return of the replicated index with Transaction Costs.

Figure 7. Rolling Correlation (n = 21) between the SPXTR Index and PCLS Index with TC



Note: Figure presents the coefficient for rolling Pearson Correlation between the vector of the replicated index daily returns and the vector of the SPXTR Index daily returns. The rolling window for the calculation is set to 21 trading days.

4.3 Index based on Non-Negative Least Squares (NNLS)

From the visual, high-level analysis of Non-Negative Least Squares portfolio for cumulative returns in Figure 8, it may seem that we are on a good track to minimize the Tracking Error. With another method we have managed to decrease the error to the SPXTR Index again. As seen on the distribution plot (Figure 9), less observations for daily returns are in the right and left tail of the distribution, and the kurtosis is higher than for PCOR and PCLS indices. Not only we have managed to decrease the excess return, but the rolling correlation (Figure 10) is much more stable, and the outliers are observations which are falling below the value of 50%, which is a substantial improvement. From the analysis of Table 3, we notice that the Absolute Tracking Error (aTE) is increasing after inclusion of TC, but the overall decreasing trend from the last 3 methods is promising.

Table 3. Performance Metrics for NNLS Index

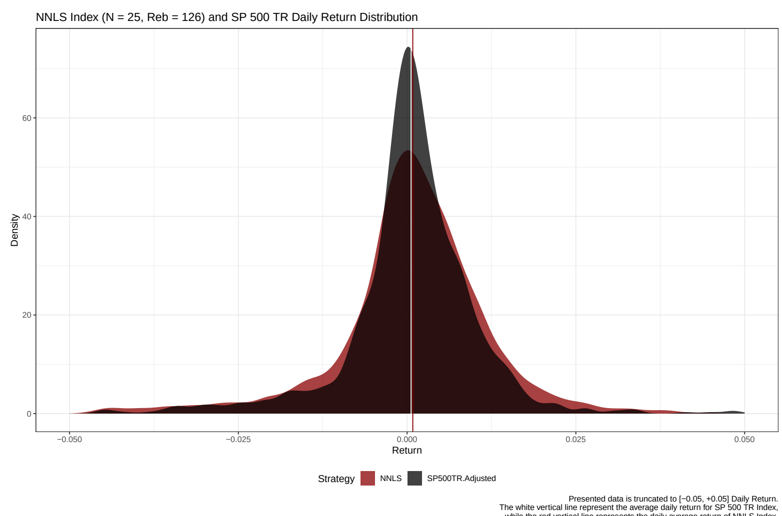
	NNLS			Benchmark ETFs		
Name	NNLS	NNLS_TC	SPX	SPY	VOO	IVV
ART	0.2054	0.1761	0.1342	0.1336	0.1337	0.1323
CRT	1.3181	1.0747	0.8759	0.8708	0.8717	0.8602
ASD	0.2235	0.2245	0.1932	0.1895	0.1930	0.1933
ASR	0.8388	0.7063	0.6070	0.6157	0.6051	0.5973
MDD	0.3382	0.3382	0.3597	0.3575	0.3620	0.3612
AIR	0.9190	0.7842	0.6944	0.7047	0.6925	0.6845
aTE	0.0728	0.0752	0.0000	0.0116	0.0088	0.0097
pTE	0.0539	0.0538	0.0000	0.0108	0.0073	0.0076
nTE	0.0481	0.0533	0.0000	0.0091	0.0069	0.0081

Note: The above table presents the performance metrics for Index based on Non-Negative Least Squares with and without Transaction Costs included in comparison to the SPXTR Index and three benchmark ETFs. Portfolio allocation is based on the coefficient value, where it rewards higher coefficients. Weights are scaled to sum up to 100

Figure 8. Cumulative Return of the SPXTR Index and NNLS Index with and without TC

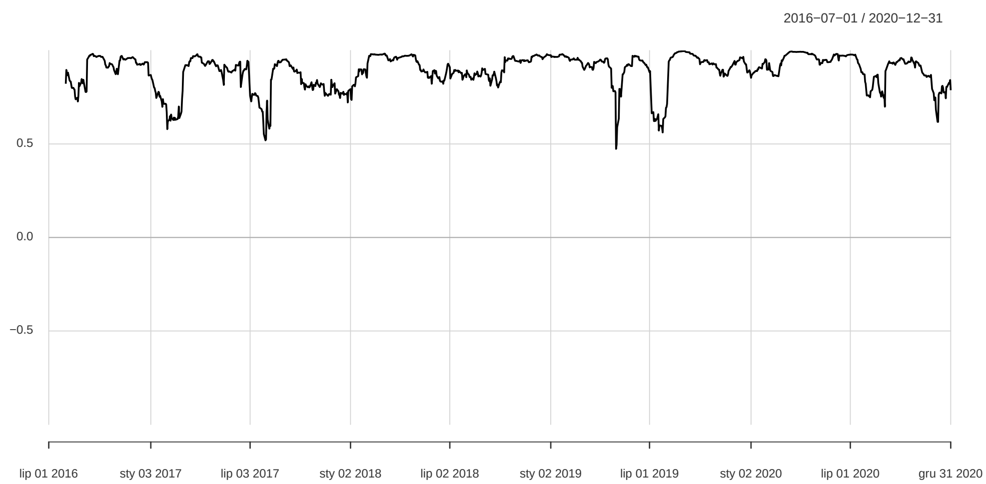
Note: Figure presents the cumulative return for the Index without Transaction Costs, the Index with Transaction Costs and the SPXTR Index. Equity line is calculated by taking the cumulative product of daily returns.

Figure 9. Return Distibutions of the SPXTR Index and NNLS Index



Note: Figure presents the distribution of daily returns for the replicated index with Transaction Costs and the SPXTR Index. Presented data is truncated to [-0.05, +0.05] daily returns. The white vertical line represents the average daily return for the SPXTR Index, while the red vertical line represents the daily average return of the replicated index with Transaction Costs.

Figure 10. Rolling Correlation (n = 21) between the SPXTR Index and NNLS Index with TC



Note: Figure presents the coefficient for rolling Pearson Correlation between the vector of the replicated index daily returns and the vector of the SPXTR Index daily returns. The rolling window for the calculation is set to 21 trading days.

4.4 Index based on Market Beta (BETA)

The results obtained by calculating the Beta coefficients for asset allocation are surprising from the perspective of earlier performance. In the previous methods we have substantially outperformed the SPXTR Index, however now we seem to be underperforming even before the TC inclusion. This might be due to the left skew in the return distribution as visible on the appropriate Figure (12). On the positive note, we see that the TE is further decreased to more satisfying levels (Table 4) and the overall performance and cumulative return is much closer to the SPXTR Index (Figure 11). The rolling correlation in Figure 13 has certainly improved as well, but not as significantly as we would like to achieve.

Table 4. Performance Metrics for BETA Index

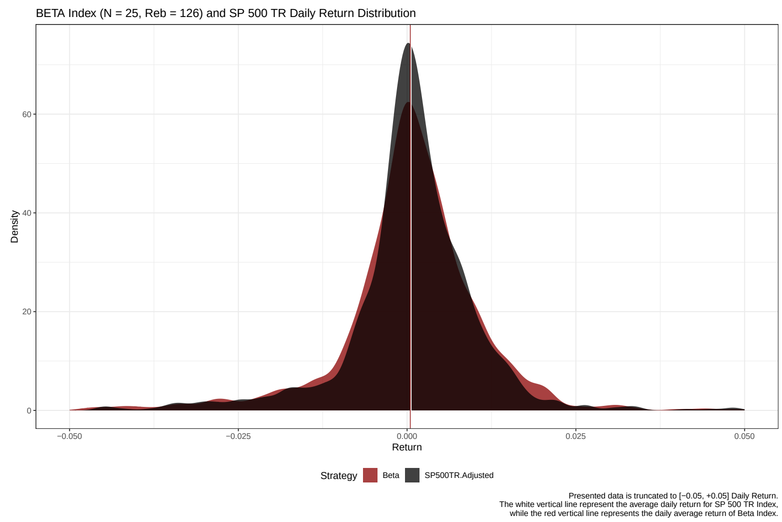
	Market Beta			Benchmark ETFs		
Name	BETA	BETA_TC	SPX	SPY	VOO	IVV
ART	0.1049	0.0916	0.1342	0.1336	0.1337	0.1323
CRT	0.5663	0.4832	0.8759	0.8708	0.8717	0.8602
ASD	0.2175	0.2177	0.1932	0.1895	0.1930	0.1933
ASR	0.4064	0.3459	0.6070	0.6157	0.6051	0.5973
MDD	0.3829	0.3829	0.3597	0.3575	0.3620	0.3612
AIR	0.4820	0.4205	0.6944	0.7047	0.6925	0.6845
aTE	0.0486	0.0493	0.0000	0.0116	0.0088	0.0097
pTE	0.0344	0.0343	0.0000	0.0108	0.0073	0.0076
nTE	0.0323	0.0337	0.0000	0.0091	0.0069	0.0081

Note: The above table presents the performance metrics for Index based on Beta Coefficient with and without Transaction Costs included in comparison to the SPXTR Index and three benchmark ETFs. Portfolio allocation is based on the coefficient value, where it rewards coefficient close to the value of 1, by setting the weight for each asset as the inverted coefficient value. Weights are scaled to sum up to 100

Figure 11. Cumulative Return of the SPXTR Index and BETA Index with and without TC

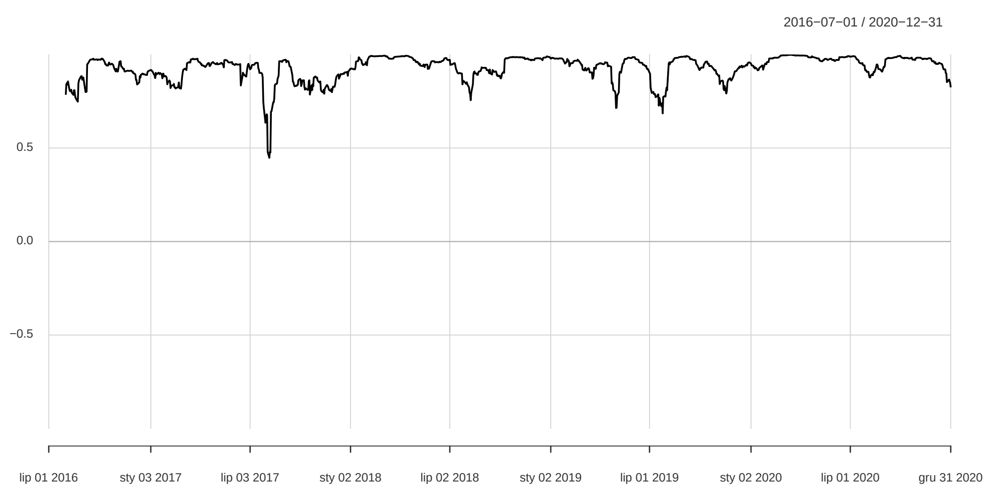
Note: Figure presents the cumulative return for the Index without Transaction Costs, the Index with Transaction Costs and the SPXTR Index. Equity line is calculated by taking the cumulative product of daily returns.

Figure 12. Return Distibutions of the SPXTR Index and BETA Index



Note: Figure presents the distribution of daily returns for the replicated index with Transaction Costs and the SPXTR Index. Presented data is truncated to [-0.05, +0.05] daily returns. The white vertical line represents the average daily return for the SPXTR Index, while the red vertical line represents the daily average return of the replicated index with Transaction Costs.

Figure 13. Rolling Correlation (n = 21) between the SPXTR Index and BETA Index with TC



Note: Figure presents the coefficient for rolling Pearson Correlation between the vector of the replicated index daily returns and the vector of the SPXTR Index daily returns. The rolling window for the calculation is set to 21 trading days.

4.5 Index based on Dynamic Time Warping (DTWA)

Dynamic Time Warping approach presumably delivers the best results in comparison to all Simple methods we used. We seem to have a periods of outperformance and underperformance as seen on the cumulative plots (Figure 14), which with enhancement methods applied as well as with the positive excess return accumulation might be able to deliver satisfactory results, with high probability. As expected, the TC inclusion is increasing the underperformance as seen from the cumulative return plots and Table 5, and in fact the strategy is no longer profitable, but we need to remember that the true alpha generating methods are still to be applied. As of now, what is interesting to us is the level of TE that we can obtain with the Simple Replications. And from the rolling correlation perspective (Figure 16) it seems that now, we are on a satisfactory level. The TE is considerably lower than with e.g. PCOR Index, and currently is only 3-4 times bigger than the TE of benchmark ETFs. From the perspective of returns distribution as seen in Figure 15, we have obtained the most similar result so far.

Table 5. Performance Metrics for DTWA Index

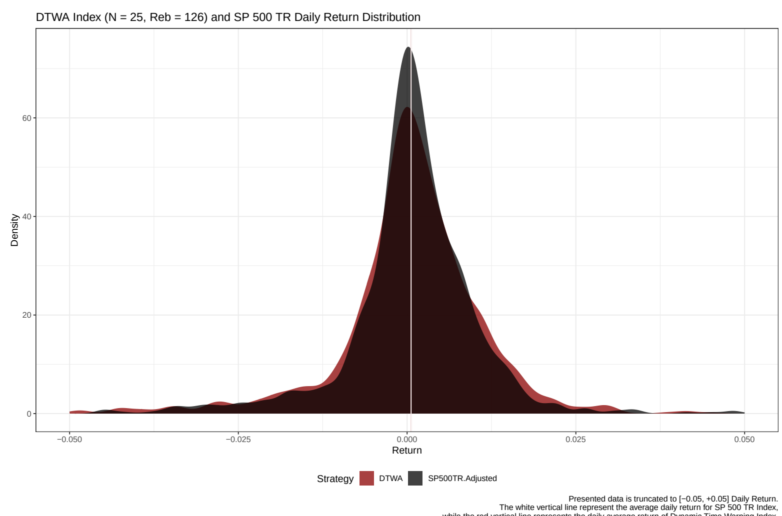
	DTW			Benchmark ETFs		
Name	DTWA	DTWA_TC	SPX	SPY	VOO	IVV
ART	0.1314	0.1205	0.1342	0.1336	0.1337	0.1323
CRT	0.7428	0.6688	0.8759	0.8708	0.8717	0.8602
ASD	0.2196	0.2198	0.1932	0.1895	0.1930	0.1933
ASR	0.5216	0.4725	0.6070	0.6157	0.6051	0.5973
MDD	0.3715	0.3715	0.3597	0.3575	0.3620	0.3612
AIR	0.5983	0.5484	0.6944	0.7047	0.6925	0.6845
aTE	0.0472	0.0478	0.0000	0.0116	0.0088	0.0097
pTE	0.0343	0.0342	0.0000	0.0108	0.0073	0.0076
nTE	0.0313	0.0326	0.0000	0.0091	0.0069	0.0081

Note: The above table presents the performance metrics for Index based on Dynamic Time Warping with and without Transaction Costs included in comparison to the SPXTR Index and three benchmark ETFs. Portfolio allocation is based on the distance between the SPXTR Index and particular asset, where it rewards the assets with smaller distance, by setting the weight for each asset as the inverted distance. Weights are scaled to sum up to 100

Figure 14. Cumulative Return of the SPXTR Index and DTWA Index with and without TC

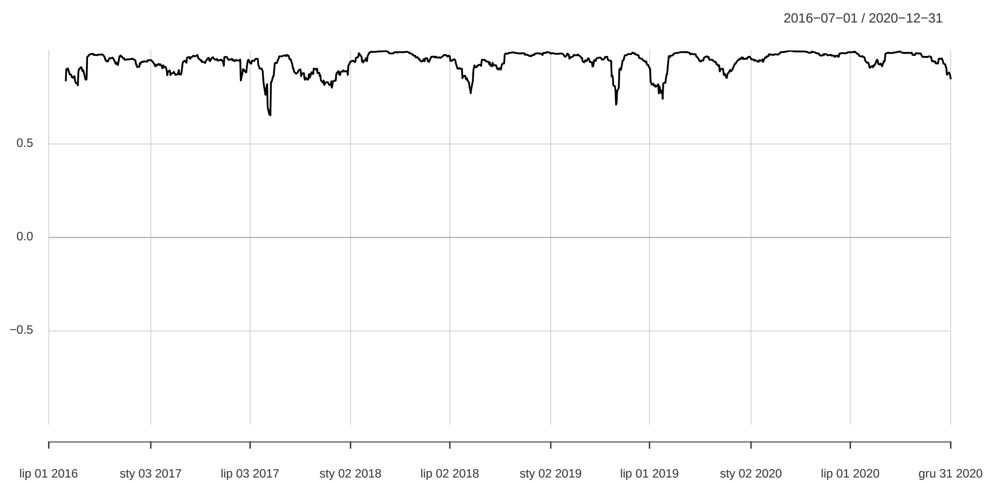
Note: Figure presents the cumulative return for the Index without Transaction Costs, the Index with Transaction Costs and the SPXTR Index. Equity line is calculated by taking the cumulative product of daily returns.

Figure 15. Return Distibutions of the SPXTR Index and DTWA Index



Note: Figure presents the distribution of daily returns for the replicated index with Transaction Costs and the SPXTR Index. Presented data is truncated to [-0.05, +0.05] daily returns. The white vertical line represents the average daily return for the SPXTR Index, while the red vertical line represents the daily average return of the replicated index with Transaction Costs.

Figure 16. Rolling Correlation (n = 21) between the SPXTR Index and DTWA Index with TC



Note: Figure presents the coefficient for rolling Pearson Correlation between the vector of the replicated index daily returns and the vector of the SPXTR Index daily returns. The rolling window for the calculation is set to 21 trading days.

4.6 Simple Index Replication Research Questions

Below we present the answers for Research Questions (1) and (2).

For RQ1, we present the p-value of *t.test* for the difference of the means under the alternative, that the true difference is greater than zero. We test under 95% confidence interval and alpha significance level of 5%. In other words, if the p-value for the test is greater than 0.05, we accept the null hypothesis of no statistically significant difference between the means of the replicated index and the SPXTR Index daily return distribution.

For RQ2 we check whether Absolute Tracking Error to the SPXTR Index is within our pre-defined limits, that is [0.0044, 0.0233]. Exact numbers of Absolute, Positive and Negative Tracking Errors are included in the Tables above, where present the overall performance of each method, while in this Table we verify whether the value is within limit or not.

Table 6. Simple Index Replication Research Questions

	Without TC		With TC	
Method	RQ1	RQ2	RQ1.TC	RQ2.TC
PCOR	0.3061	Not Met	0.4256	Not Met
PCLS	0.6445	Not Met	0.8642	Not Met
NNLS	0.6191	Not Met	0.7522	Not Met
BETA	0.8741	Not Met	0.8035	Not Met
DTWA	0.9819	Not Met	0.9609	Not Met

Note: For Research Question 1 (RQ1) we present the p-value of t.test for the difference of the means between two vector of returns. The null hypothesis states, that there is no difference. For Research Question 2 (RQ2) we present the binary outcome of the test, whether a certain method delivers Absolute Tracking Error within the predefined limits. If not, the result of the test is 'Not Met' and 'Met', if otherwise. The TE limit is defined as $[0.5 * TE_L; 2.0 * TE_H]$ where TE_L is the lowest and TE_H is the highest TE among benchmarks.

From Table 6 we see, that all Simple methods have passed the test for the daily return distribution similar to the SPXTR Index with statistical significance (RQ1). The most significant result is the one from Dynamic Time Warping method. None of the methods passed the test of having the Absolute Tracking Error within the allowed limit we defined (RQ2).

5 Enhanced Index Replication

In this section, we present the result of enhancement methods applied to the best Simple Index, namely the DTWA Index, which performed the best during the last Chapter. In the Simple Index Replication, we were interested in bringing the TE and return distribution as close to the SPXTR Index as possible. Indeed, the DTWA Index scored the lowest aTE, pTE and nTE. Additionally, the p-value for the *t.test* is the highest for DTWA Index, which indicates that the lack of difference between return distribution (mean) is the most significant.

5.1 Simple Index with ERC

Let us first see, what would be the outcome of the *excess return cushion* (ERC) applied to the DTWA Index without the enhancement. Now, we apply TC to each experiment and we are no longer interested in the performance of indices without TC included.

Figure 17. Simple DTWA Index with ERC and the SPXTR Index

Note: Figure presents the cumulative return for the DTWA Index with Transaction Costs, DTWA Index with Transaction Costs and ERC and without Dividends, DTWA Index with Transaction Costs and ERC and with Dividends and the SPXTR Index. Equity line is calculated by taking the cumulative product of daily returns.

As expected and discussed earlier, the ERC applied to the Simple Index will decrease the overall performance and returns, due to restricting the outperformance, before introducing the enhancement methods that aim to deliver better alpha. That is why, we aim to increase the positive excess return in the overall return distribution. As of now, we just limited the possibility of overperforming, which results in lower returns compared to the original Simple Index. Question is, how the performance will look after applying the enhancement methods. From Table 7, we can analyse the results presented above based on our performance measurements. In the end, the ERC application to the Simple Index is not a profitable idea, and as we outlined previously, next we will compare enhanced methods with ERC to the Simple Index, to see, whether they provide a measurable improvement. It is important to note, that we should consider ERC only combined with enhancement methods.

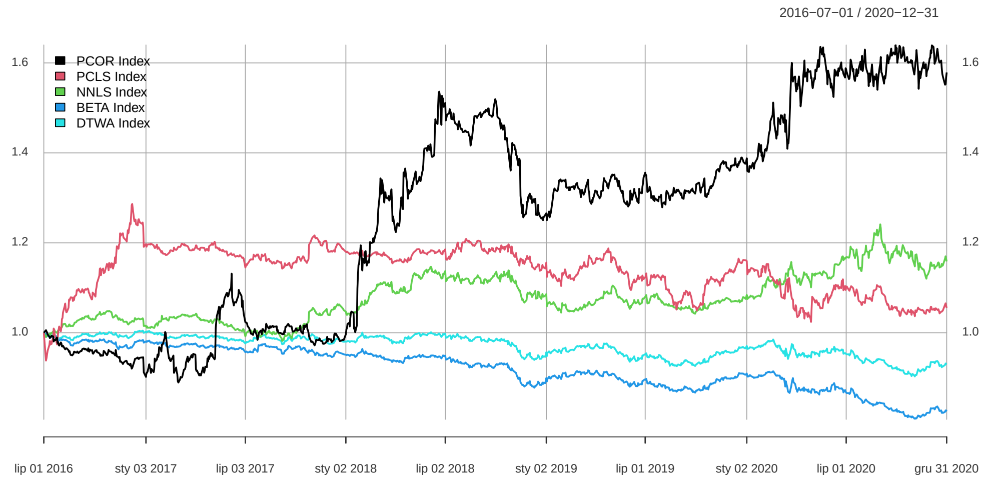
Table 7. Simple Index with ERC Performance for DTWA Index

		ERC	ERC + Div	
Name	DTWA_Simple	DTWA_Simple.1	DTWA_Simple.2	SPXTR
ART	0.1205	0.1177	0.1171	0.1382
CRT	0.6688	0.6501	0.6461	0.7907
ASD	0.2198	0.1975	0.2001	0.1966
ASR	0.4725	0.5119	0.5023	0.6169
MDD	0.3715	0.3597	0.3702	0.3597
AIR	0.5484	0.5961	0.5854	0.7030
aTE	0.0478	0.0094	0.0278	0.0000
pTE	0.0342	0.0000	0.0254	0.0000
nTE	0.0326	0.0094	0.0134	0.0000

Note: The above table presents the performance metrics for Simple Index based on Dynamic Time Warping with Transaction Costs included with and without ERC applied for the option with and without dividend payment. The pTE is higher than zero for the options with dividend payment, as the payment is not excluded from the Tracking Error calculation, therefore on each rebalancing day, we observe an abnormal return, if the dividend is paid out. The Absolute Tracking Error (aTE) can be higher as well for the solutions with dividend payment, as we reduce our accumulated ERC to zero along with the event of dividend payout. Additionally, the SPXTR Index is presented for comparison.

Before we proceed to the enhancement methods with and without ERC applied to DTWA Index, which we found to be the best suited for the index-tracking purposes, we can examine the effect that ERC could have on the replicated index with the best performance, namely the PCOR Index. As seen in Figure 18, the relative performance measured against SPXTR Index is the best for the Partial Correlation replication method.

Figure 18. Simple Index Relative Performance to SPXTR Index



Note: The above figure presents the cumulative relative performance to SPXTR Index for each Simple Index with Transaction Costs included. Equity line is calculated by taking the cumulative product of daily returns.

From Figure 19 and Table 8, we see that the overperformance on the Simple Index is not always working as desired with the ERC applied. That is due to a fact, that the ERC success is highly dependend on the beginning of the analyzed period. In other words, when we saw a period of underperformance for the first year of the analysis, the ERC without dividend payment is shifted permanently relative to SPXTR Index and is not able to cover the underperforming period itself. Only with the dividend inclusion, we can fill this gap and therefore deliver a substantial alpha to the investor. Nevertheless, from the examination of the DTWA Index, which is the closest in terms of tracking and the PCOR Index, which delivered the highest overperformance, we believe, that for the Skewness and Kurtosis enhancement as well as for the application of ERC, the replicated index, which tracks the SPXTR Index the closest is the most suitable.

Figure 19. Simple PCOR Index with ERC and the SPXTR Index

Note: Figure presents the cumulative return for the PCOR Index with Transaction Costs, PCOR Index with Transaction Costs and ERC and without Dividends, PCOR Index with Transaction Costs and ERC and with Dividends and the SPXTR Index. Equity line is calculated by taking the cumulative product of daily returns.

Table 8. Simple Index with ERC Performance for PCOR Index

		ERC	ERC + Div	
Name	PCOR_Simple	PCOR_Simple.1	PCOR_Simple.2	SPXTR
ART	0.2598	0.1099	0.2500	0.1382
CRT	1.8270	0.5990	1.7293	0.7907
ASD	0.2825	0.1967	0.3017	0.1966
ASR	0.8533	0.4748	0.7670	0.6169
MDD	0.3608	0.3597	0.3597	0.3597
AIR	0.9197	0.5588	0.8287	0.7030
aTE	0.1686	0.0117	0.2278	0.0000
pTE	0.1340	0.0000	0.2362	0.0000
nTE	0.1151	0.0117	0.0313	0.0000

Note: The above table presents the performance metrics for Simple Index based on Partial Correlation with Transaction Costs included with and without ERC applied for the option with and without dividend payment. The pTE is higher than zero for the options with dividend payment, as the payment is not excluded from the Tracking Error calculation, therefore on each rebalancing day, we observe an abnormal return, if the dividend is paid out. The Absolute Tracking Error (aTE) can be higher as well for the solutions with dividend payment, as we reduce our accumulated ERC to zero along with the event of dividend payout. Additionally, the SPXTR Index is presented for comparison.

5.2 Kurtosis and Skewness Enhanced Index without ERC

Before we apply the ERC to the enhancement methods it is worth considering and analysing the effect of the unrestricted Kurtosis and Skewness application to the returns and performance metrics. Figure 20 and Table 9 present these results.

Figure 20. Kurtosis and Skewness Enhanced Index without ERC and the SPXTR Index

Note: Figure presents the cumulative return for the DTWA Index with Transaction Costs, DTWA Index with Transaction Costs and Kurtosis Enhancement without ERC, DTWA Index with Transaction Costs Skewness Enhancement without ERC and the SPXTR Index. Equity line is calculated by taking the cumulative product of daily returns.

Table 9. Kurtosis and Skewness Enhanced Index without ERC Performance for DTWA Index

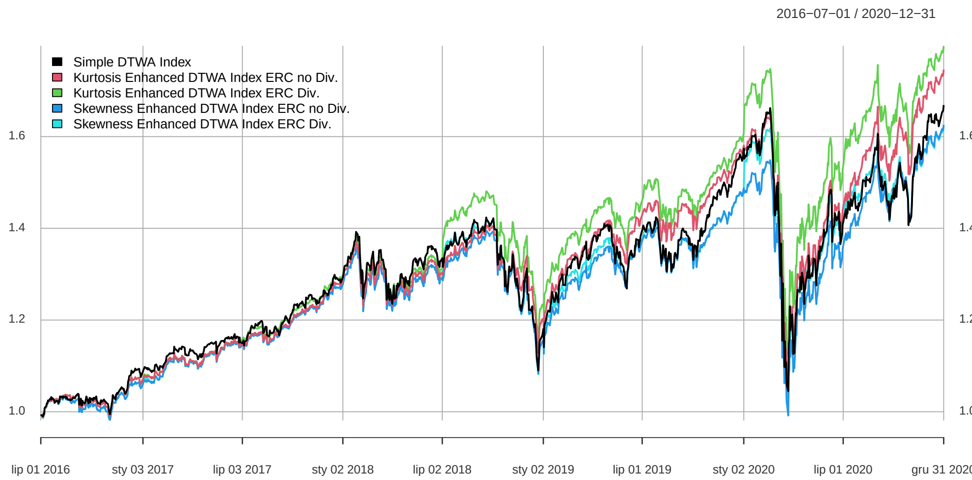
	Simple	Kurtosis Enhanced	Skewness Enhanced	
Name	DTWA_Simple	DTWA_Kurtosis	DTWA_Skewness	SPX
ART	0.1205	0.1439	0.1176	0.1382
CRT	0.6688	0.8309	0.6494	0.7907
ASD	0.2198	0.2127	0.2198	0.1966
ASR	0.4725	0.5963	0.4594	0.6169
MDD	0.3715	0.3526	0.3656	0.3597
AIR	0.5484	0.6764	0.5351	0.7030
aTE	0.0478	0.0533	0.0534	0.0000
pTE	0.0342	0.0368	0.0375	0.0000
nTE	0.0326	0.0367	0.0369	0.0000

Note: The above table presents the performance metrics for Simple Index based on Dynamic Time Warping with Transaction Costs included with Skewness and Kurtosis enhancements without ERC applied. Additionally, the SPXTR Index is presented for comparison.

5.3 Kurtosis and Skewness Enhanced Index with ERC

Now, we apply the ERC to enhancement methods, that together are aiming to provide additional positive excess return. From the Figure 21 we notice, that the Kurtosis correction significantly increases the overall performance, while the Skewness seems to be underperforming. We present the results for the strategy with and without dividend payment

Figure 21. Kurtosis and Skewness Enhanced Index with ERC with and without Dividend and the Simple Index



Note: Figure presents the cumulative return for the DTWA Index with Transaction Costs (DTWA TC), DTWA TC with Kurtosis Enhancement with and without Dividends and DTWA TC with Skewness Enhancement with and without Dividends. Equity line is calculated by taking the cumulative product of daily returns.

In Table 10 we present the comparison between Simple Index based on Dynamic Time Warping with TC included and four additional enhanced indices. Each enhancement is presented with and without cyclical dividend payment. As mentioned at the beginning of this Chapter, all indices are with TC included. All enhanced indices have the *excess return cushion* (ERC) already included, therefore for indices without dividend payment, the Positive Tracking Error (pTE) is zero. Additionally, we can notice a significant reduction in the Negative Tracking Error (nTE). What is more, the decrease in nTE in comparison to decrease in aTE is not proportional. This means that indeed the enhancement methods provide more exposure to pTE at the benefit of decreasing the nTE.

Table 10. Kurtosis and Skewness Enhanced Index with ERC Performance for DTWA Index

	Simple	Kurtosis Enhanced		Skewness Enhanced		
Name	DTWA_Simple	DTWA	DTWA_Div	DTWA.1	DTWA_Div.1	SPX
ART	0.1205	0.1318	0.1392	0.1141	0.1135	0.1382
CRT	0.6688	0.7459	0.7973	0.6259	0.6225	0.7907
ASD	0.2198	0.1967	0.1998	0.1976	0.1999	0.1966
ASR	0.4725	0.5847	0.6115	0.4934	0.4851	0.6169
MDD	0.3715	0.3597	0.3597	0.3597	0.3640	0.3597
AIR	0.5484	0.6703	0.6963	0.5773	0.5680	0.7030
aTE	0.0478	0.0050	0.0363	0.0106	0.0327	0.0000
pTE	0.0342	0.0000	0.0357	0.0000	0.0302	0.0000
nTE	0.0326	0.0050	0.0116	0.0107	0.0150	0.0000

Note: The above table presents the performance metrics for Index based on Dynamic Time Warping with Transaction Costs included and for the same method with Transactions Costs and two enhancement methods (Kurtosis and Skewness). Enhancement methods results are presented for the option with and without dividend payment. The pTE is higher than zero for the options with dividend payment, as the payment is not excluded from the Tracking Error calculation, therefore on each rebalancing day, we observe an abnormal return, if the dividend is paid out. The Absolute Tracking Error (aTE) can be higher as well for the solutions with dividend payment, as we reduce our accumulated ERC to zero along with the event of dividend payout. Additionally, the SPXTR Index is presented for comparison.

5.4 Enhanced Index Replication Research Questions

We now review the initial Research Questions for the enhancement part. Let us bring them once again, to reiterate:

RQ3: *With the enhancement applied (Kurtosis or Skewness), it is possible to create a replicated index using N components that without ERC will have a different return distribution in comparison to the SPXTR Index, with statistical significance, applying the statistical tests mentioned in H1.*

RQ4: *With the enhancement applied (Kurtosis or Skewness), it is possible to create a replicated index using N components that without ERC will have a higher Positive Tracking Error than benchmark ETFs in absolute terms.*

RQ5: *While the above (RQ4) is true, the replicated index with ERC will have the Absolute TE within the allowed limits.*

Table 11. Enhanced Index Replication Research Questions

	Without Dividend			With Dividend		
Enhancement	RQ3	RQ4	RQ5	RQ3.Div	RQ4.Div	RQ5.Div
Kurtosis	0.9246	Met	Met	0.9246	Met	Not Met
Skewness	0.9455	Met	Met	0.9455	Met	Not Met

Note: For Research Question 3 (RQ3) we present the p-value of t.test for the difference of the means between two vector of returns. The null hypothesis states, that there is no difference. For Research Question 3 (RQ3) and Research Question 4 (RQ4) we present the binary outcome of the test, whether a certain method is compliant with the statement provided in Research Questions. If not, the result of the test is 'Not Met' and 'Met', if otherwise.

We see that all enhancement methods did not manage to fulfill the 3^rd Research Question, as in every case, we cannot reject the null hypothesis in favour of the alternative hypothesis. There is no statistically significant difference between the mean daily return of neither of the Enhanced indices and the SPXTR Index. On the other hand, we can positively answer the RQ4. In every case, prior to applying the ERC, we overweighted the Positive Tracking Error. Last but not least, the assumption under RQ5 works without dividend payments.

In the end, it means that we are able to create a replicated the SPXTR Index, that with baseline Dynamic Time Warping asset allocation and either of the enhancement method applied, will have higher Positive TE than benchmark ETFs, while having the Absolute Tracking Error within the defined limit (if we do not pay dividends). However, such replicated index on average will be indistinguishable from the SPXTR Index.

Given the better performance of Kurtosis enhancement method, we proceed to the Main Hypothesis evaluation, which we defined as:

H1: With enhancement methods applied, it is possible to create a synthetic index of N components[4] with absolute Tracking Error similar to benchmark ETFs[5]. Such Index will have a return distribution significantly[6] skewed to the right in comparison to the SPXTR Index, therefore positive deviations will be more frequent than negative ones.

Given all the evidence on the training sample, we have to reject the Main Hypothesis, as we did

[4] The number N is smaller than the total number of S&P 500 Index constituents in every holding period.

[5] The TE limit is defined as $[0.5 * TE_L; 2.0 * TE_H]$ where TE_L is the lowest, TE_H is the highest TE among benchmark ETFs.

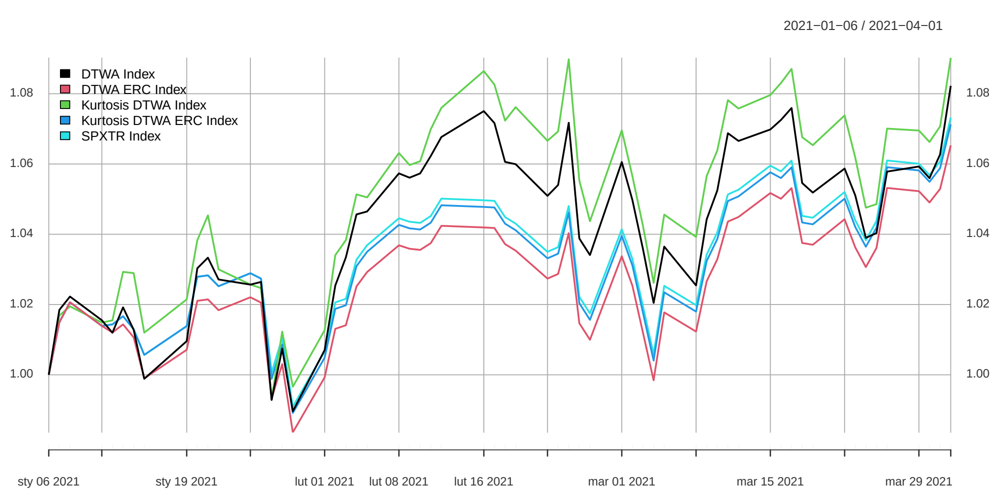
[6] The statistical significance is verified with a two sample $t - test$ for difference of means with 95% confidence interval.

not manage to skew the return distribution enough to the positive side, in order to make it different from the SPXTR Index return distribution. However, we can partially confirm the rest of the Main Hypothesis, especially its first part because with enhancement methods applied, it is possible to create a synthetic index of N components with absolute Tracking Error similar to benchmark ETFs. Now, we proceed to the out-of-sample and out-of-time evaluation of Kurtosis Enhanced Dynamic Time Warping Index and Simple Dynamic Time Warping Index, together with the SPXTR Index and benchmark ETFs performance.

6 Final Results on Out-of-Sample Data

We ran the test for the first quarter of 2021 (Q1 2021), which consists of the data not yet seen by algorithms. We sourced data from 2020-07-01 to be able to find the weights for 25 assets on a 126-day rebalancing period. For the first out-of-sample trading day on 4ˆth January 2021, we allocated the assets according to the Dynamic Time Warping algorithm and the DTWA Index with Kurtosis Enhancement, as this method of enhancement turned out to be performing best on the in-sample data. Below Figure 22 presents the results. We do not include solution with dividend payment, as our out-of-sample period is shorter than rebalancing period. We additionally present the results of Kurtosis enhancement without ERC, as we believe it is important due to the fact, that the testing period is too short to use accumulated and not used ERC in a form of dividend payout.

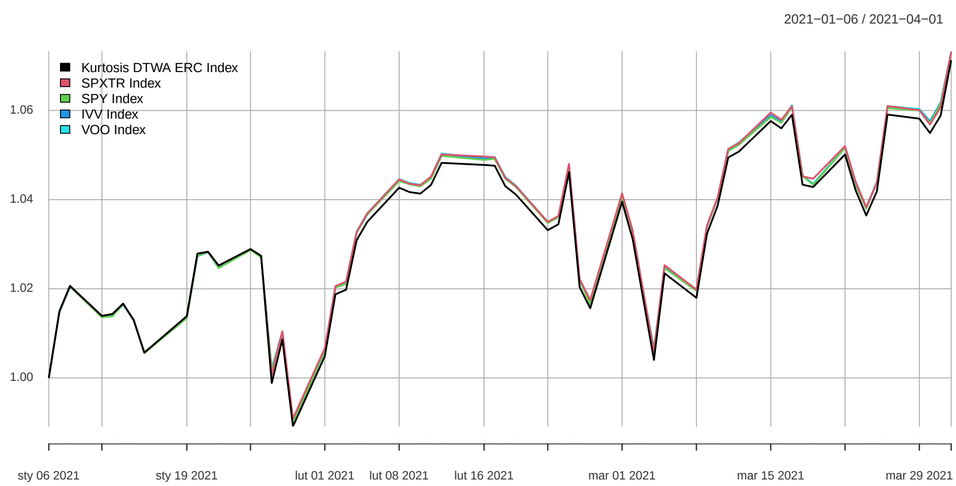
Figure 22. Test Results for DTWA Index and the SPXTR Index



Note: Figure presents the cumulative return for the DTWA Index with Transaction Costs (DTWA TC), DTWA TC with Kurtosis Enhancement with and without ERC. Equity line is calculated by taking the cumulative product of daily returns.

We see that the DTWA Index is successful even on the data never seen before. The DTWA Index with additional Kurtosis Enhancement is generating a strong overperformance in the first quarter of 2021. Once we apply the *excess return cushion* (ERC), the performance from the visual inspection is not distinguishable from benchmark ETFs. Below Figure 23 and Table 12 provide a summary for the test results.

Figure 23. Test Results for DTWA Index with ERC and Kurtosis Enhancement and Benchmark ETFs



Note: Figure presents the cumulative return for the DTWA TC with Kurtosis Enhancement with ERC, the SPXTR Index and three benchmark ETFs. Equity line is calculated by taking the cumulative product of daily returns.

Table 12. Table with Final Test Results for DTWA Index

	Dynamic Time Warping					Benchmark ETFs		
Name	Simple	Simple.ERC	Kurtosis	Kurtosis.ERC	SPXTR	SPY	IVV	VOO
ART	0.3937	0.3043	0.4371	0.3353	0.3455	0.3436	0.3442	0.3444
CRT	0.0822	0.0653	0.0902	0.0713	0.0732	0.0728	0.0730	0.0730
ASD	0.1932	0.1593	0.1999	0.1587	0.1574	0.1558	0.1554	0.1555
ASR	1.9306	1.7887	2.0801	1.9873	2.0675	2.0765	2.0866	2.0858
MDD	0.0508	0.0422	0.0584	0.0422	0.0422	0.0415	0.0417	0.0416
AIR	2.0378	1.9105	2.1870	2.1124	2.1946	2.2048	2.2152	2.2143
aTE	0.0591	0.0101	0.0759	0.0036	0.0000	0.0079	0.0079	0.0084
pTE	0.0340	0.0000	0.0494	0.0000	0.0000	0.0043	0.0044	0.0047
nTE	0.0346	0.0101	0.0471	0.0036	0.0000	0.0060	0.0062	0.0070

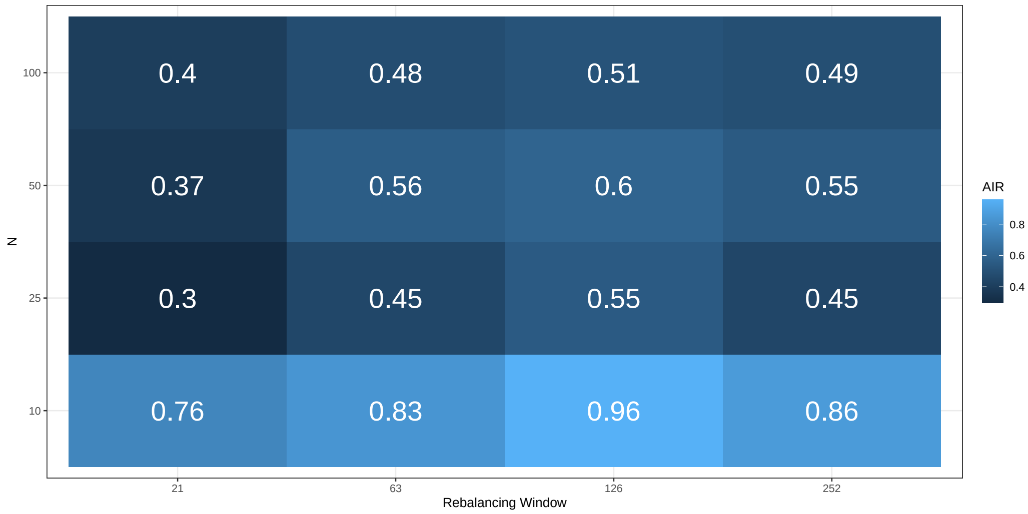
Note: The above table presents the performance metrics on out-of-sample testing period for Index based on Dynamic Time Warping (DTWA Index) with and without ERC, as well as for DTWA Index with Kurtosis Enhancement with and without ERC in comparison to the SPXTR Index and three benchmark ETFs.

We see again that indeed the ERC implementation helps keep up with the SPXTR Index as well as strongly reduce the Negative TE (the Positive is naturally 0). The final DTWA Index with ERC and Kurtosis Enhancement is on pair with the most popular ETFs that track and provide exposure to the S&P 500 Index.

7 Sensitivity Analysis

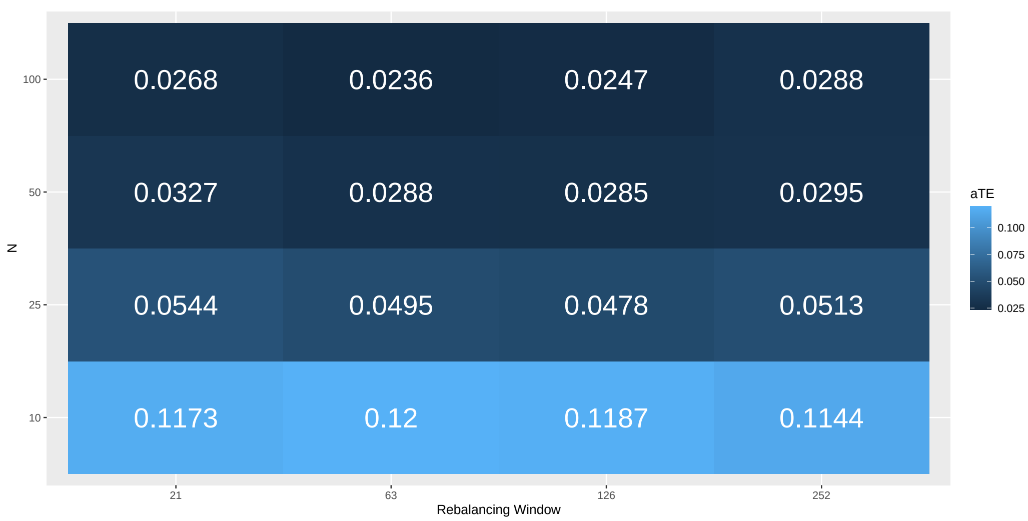
In this second to last Chapter, we explore the robustness of the selected method. We know from the testing performance, that the results are satisfying on the out-of-sample results, but how lucky are we with picking the initial $N = 25$ and $Reb = 126$ for our initial algorithm parameters? We check the performance of Dynamic Time Warping Index for new Reb and N parameters. Our new set of parameters is:

Figure 24. Sensitivity Analysis for Annualized Information Ratio



Note: Figure presents the results of sensitivity analysis on in-sample training data for 16 combinations of N (Number of Assets in Portfolio) and Reb (Length of Rebalancing Period, in trading days).

Figure 25. Sensitivity Analysis for Annualized Absolute Tracking Error



Note: Figure presents the results of sensitivity analysis on in-sample training data for 16 combinations of N (Number of Assets in Portfolio) and Reb (Length of Rebalancing Period, in trading days).

Figures 24 and 25 show that there is a trade-off between the performance (in a sense of positive excess return or AIR) and Tracking Error. The higher the N, the lower the TE and lower the annualized risk-adjusted returns. With the increase of N, we notice the opposite reaction. The length of the rebalancing window has no clear directional impact on neither of the metrics.

8 Conclusion

In this research paper, we aimed to implement an index-tracking mechanism with a limited number of assets. The replicated index was supposed to be comparable with benchmark ETFs, namely SPY, VOO and IVV. Moreover, with the invention of *excess return cushion* we wanted to capture the positive excess return and to hedge the risk of underperformance. As we found in the literature related to the topic, the Smart Beta funds lack the consistent pattern of overperformance in the comparison to their simple, index-tracking counterparts. Nevertheless, we also found the evidence that the excess return or "beating the benchmark" is an important goal for both individual and institutional investors. That said, we believe that our approach is a "best of both worlds" solution. We focus on the tracking behaviour, but with the enhancement method applied, we give a chance - not a promise - for additional abnormal returns, which can be paid out in the form of a dividend.

We used S&P 500 Total Return as the Index for replication. The comparison between different solutions were based on training data from the beginning of 2016 to the end of 2020, while the first quarter of 2021 was used for out-of-sample testing. For the Simple Index Replication we used Partial Correlation, Non-Negative Least Squares, Beta Coefficient and Dynamic Time Warping as our methods for asset allocation of 25 equities on a 126-day rebalancing window. We found empirical evidence that the Dynamic Time Warping is best suited to mimic the SPXTR Index during the analyzed period. We believe that it is due to the fact that this method is the only one with a sequential character. In other words, in contrast to other methods, it is working on a time series, rather than just a vector. This is important as time series data is considered a sequential type and this additional filter provided in Dynamic Time Warping, that analyzes the sequential patterns has turned out to be effective.

For the Enhanced Index Replication, we introduced two methods, one based on the Kurtosis and/or the Skewness of individual asset return distribution in comparison to the SPXTR Index and the second one based on adding the *excess return cushion*. We found that the ERC enhancement performs better than the Kurtosis enhancement which performs better than the Skewness one (Table 9 and 10).

During the empirical research, we confirmed that it is possible to create a replicated index with 25 components that will have a similar return distribution as the SPXTR Index (RQ1). On the other hand, with just Simple Index Replication it is not possible to achieve Tracking Error within the limits we assumed as comparable with benchmark ETFs (RQ2). Moreover, with the Enhanced Index Replication, we failed to skew the daily return distribution enough to say that there is a difference in mean return between the SPXTR Index and our replicated index with enhancement methods applied (RQ3). However, we can confirm, that with the enhancement methods in place we can provide a higher Positive Tracking Error that benchmark ETFs (RQ4) and with *excess return cushion* implemented, the Absolute Tracking Error stays within the defined limit, and our Enhanced Index is comparable with benchmark ETFs (RQ5), but not for each applied method. Given all the empirical evidence, we had to reject the main hypothesis of the research, as it assumed - among others - that we would be able to provide return distribution significantly different from the SPXTR Index. Nevertheless, our research showed that at the same time we have no grounds for rejection the first part of Main Hypothesis which means that we with enhancement methods applied, it is possible to create a synthetic index of N components with absolute Tracking Error similar to benchmark ETFs.

This research certainly provides a potential for improvement and further extensions. Based on our learnings, there is still a possibility of providing more stable and more robust enhancement methods, than those presented in the research. We believe that the use of machine learning algorithms or other sequential methods used for pattern recognition can provide a significant uplift on the positive excess return. Such methods, as Dynamic Time Warping have successfully improved the results of the tracking process in our research, therefore we strongly believe that applying them for capturing more alpha, should be a viable application.

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