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RELATIVE RISK AVERSION AND
POWER-LAW DISTRIBUTION OF
MACROECONOMIC DISASTERS

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Relative risk aversion and power-law distribution of macroeconomic disasters

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Abstract

The coefficient of relative risk aversion (CRRA) is notoriously difficult to estimate. Recently, Barro and Jin (On the size distribution of macroeconomic disasters, *Econometrica* 2011; 79(3): 434–455) have come up with a new estimation approach that fits a power-law model to the tail of distribution of macroeconomic disasters. We show that their results can be successfully replicated using a more refined power-law fitting methodology and a more comprehensive data set.

Keywords:

coefficient of relative risk aversion, power-law modelling, macroeconomic disasters, replication, robust statistics

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1. Introduction

The coefficient of relative risk aversion, γ , is often used in microeconomic and macroeconomic models. However, the range of its estimates found in the empirical literature is rather wide. Recently, [Barro and Jin \(2011\)](#) have proposed a new method of estimating γ , using a model that explains the equity premium with reference to rare macroeconomic disasters, i.e., contractions in real per capita GDP or per capita consumption of size 10% or more. Their approach relies on modeling the upper tail of the size distribution of disasters with a power-law model, extensively used in various parts of economics and other sciences ([Clauset et al., 2009](#); [Gabaix, 2009](#)). In order to improve the fit of a single power-law model, they use a double power-law model ([Reed, 2003](#)), which fits two power-law models with different shape parameters (exponents) and estimates the dividing threshold. The only purpose of using double power-law model is to estimate the lower bound to the power-law behavior of the largest disasters. However, the double power-law model was not designed as a tool for choosing the optimal threshold of this kind and its statistical reliability in this respect is unknown. For this reason, the approach of [Barro and Jin \(2011\)](#) may result in biased estimation of the power-law exponent if the chosen bound is too low and both in biased estimates and large standard errors for the estimates if the bound is too high. Moreover, the authors use the maximum likelihood estimator (MLE) for estimating the exponent, but this estimator, for the power-law and some other distributions, is very sensitive to the presence of extreme observations in the upper tail of the data ([Victoria-Feser and Ronchetti, 1994](#)). Considering that the number of largest disasters following power-law behaviour as estimated by [Barro and Jin \(2011\)](#) is very small (22 for consumption and 21 for the GDP), checking the results with robust statistical methods seems worthwhile.

In this paper, we replicate the findings of [Barro and Jin \(2011\)](#) using methods from robust statistics and income distribution analysis, which allow for robust estimation of both the lower bound to the power-law behaviour and the power-law exponent. We also use a richer data set on macroeconomic disasters introduced in [Barro and Ursúa \(2012\)](#). Finally, in contrast to the original analysis conducted in Matlab, all our estimations are performed in Stata.

2. Equity Premium, Coefficient of Relative Risk Aversion and Distribution of Macroeconomic Disasters

[Barro and Jin \(2011\)](#) use a model developed in [Barro \(2006, 2009\)](#), which links the potential for rare macroeconomic disasters with the equity-premium puzzle. The model implies that the equity premium is

$$r^e - r^f = \gamma\sigma^2 + p[\mathbb{E}(1 - b)^{-\gamma} - \mathbb{E}(1 - b)^{1-\gamma} - \mathbb{E}b], \quad (1)$$

where r^e is the expected rate of return on equity, r^f is the risk-free rate of return, γ is the coefficient of relative risk aversion, σ^2 is the variance of “normal” fluctuations in the the GDP growth rate, p is a disaster probability (defined as the number of disaster to the number of nondisaster years in a sample), and b is a disaster size (fraction of output or consumption contracted). In the simplest case, the authors

assume that the upper tail of transformed disaster size z , $z \equiv 1/(1 - b)$, follows a single power-law model defined by its density function as

$$f(z) = \alpha z_0^\alpha z^{-(\alpha+1)}, \quad (2)$$

where z_0 is a lower bound to the power-law behavior and α is the power-law exponent. Equation (1) becomes then:

$$r^e - r^f = \gamma \sigma^2 + p \left\{ \left(\frac{\alpha}{\alpha - \gamma} \right) z_0^\gamma - \left(\frac{\alpha}{\alpha + 1 - \gamma} \right) z_0^{\gamma-1} + \left(\frac{\alpha}{\alpha + 1} \right) \left(\frac{1}{z_0} \right) - 1 \right\} \quad (3)$$

if $\alpha > \gamma$. In their analysis, [Barro and Jin \(2011\)](#) consider contractions of size $b = 0.095$ and more, which translates to $z_0 = 1.105$. In their basic scenario, they also assume that the equity premium is 0.05, $\sigma = 0.02$ and calculate p to be 0.038 for consumption and 0.0383 for the GDP. They further estimate α using MLE from the long-term annual data on per capita consumption for 24 countries and per capita GDP for 36 countries (see Table 1). With an estimate of α , they are able to compute a value of γ from Equation (3). Due to the poor fit of a single power-law model to z above z_0 , the authors model z with a double power-law distribution, which is a composition of two power-law models with two different exponents over two ranges of z (for details, see [Barro and Jin \(2011\)](#)). However, the only purpose of using double power-law model is to estimate a cut-off, δ , dividing the distribution of z above z_0 into the lower ($z_0 \leq z < \delta$) and the upper part ($\delta \leq z$). The power-law exponent for the upper part, estimated using MLE, is then used in a more complicated version of Equation (3) to estimate γ . Given that the double power-law model was not introduced as a tool for reliable estimation of the lower bound on power-law behavior, it is interesting to check if other methods explicitly designed for this purpose, especially robust ones, replicate the results of [Barro and Jin \(2011\)](#).

3. Robust estimation of power-law model parameters

The problem of estimating the lower bound on power-law behaviour, z_0 , has been studied in several works. For example, [Beirlant et al. \(1996\)](#) proposed a method of estimating k , the number of observations greater than z_0 , based on the minimization of an estimate of the asymptotic mean squared error (AMSE) of MLE for power-law exponent. Recently, [Clauset et al. \(2009\)](#) have proposed a simple approach to estimate z_0 using goodness-of-fit tests. The method chooses the threshold that makes the probability distribution of the data as close as possible to the power-law model with the shape parameter estimated by a preferred method. In particular, [Clauset et al. \(2009\)](#) use the Kolmogorov-Smirnov (KS) statistic as their measure of “closeness” of the distributions, but alternative measures of goodness-of-fit could be used as well.¹

Another approach developed by [Dupuis and Victoria-Feser \(2006\)](#) has the advantage of being robust to extreme observations, which often occur in the tails of

¹Some other methods, including graphical ones, are reviewed by [Dupuis and Victoria-Feser \(2006\)](#) and [Clauset et al. \(2009\)](#).

economic distributions. Dupuis and Victoria-Feser (2006) develop their robust RC -criterion for determining z_0 by applying the so-called prediction error criterion to the power-law model treated as a regression model. Let $z_{[i]}^*, i = 1, \dots, k$ be the ordered largest k observations equal or greater to z_0 , $Y_i = \log(z_{[i]}^*/z_0), i = 1, \dots, k$, and $\hat{Y}_i = -1/\hat{\alpha} \log[(k+1-i)/(k+1)], i = 1, \dots, k$, where $\hat{\alpha}$ is an estimator of α . The variance of Y_i is

$$\hat{\sigma}_i^2 = \sum_{j=1}^i \frac{1}{\hat{\alpha}^2 (k-i+j)^2}.$$

The RC -criterion is defined as

$$RC(z_0) = \frac{1}{k} \sum_{i=1}^k \hat{w}_i^2 \left(\frac{Y_i - \hat{Y}_i}{\hat{\sigma}_i} \right)^2 + \frac{2}{k} \sum_{i=1}^k \frac{1}{\hat{\sigma}_i^2} \text{cov}[\hat{w}_i Y_i, \hat{w}_i \hat{Y}_i] - \frac{1}{k} \sum_{i=1}^k \frac{1}{\hat{\sigma}_i^2} \text{var}[\hat{w}_i Y_i], \quad (4)$$

where $0 \leq \hat{w}_i \leq 1$ is a weight of the i^{th} observation, if α was estimated by a robust estimator. Bias-corrected weighted maximum likelihood estimator (WMLE) is used as a robust estimator for α (Dupuis and Victoria-Feser, 2006). The weights w_i downweight observations, which are outliers for the power-law model fitted with the WMLE. For MLE, all weights are equal to 1. Suitable expressions for covariance and variance terms in Equation 4 are provided by Dupuis and Victoria-Feser (2006). The RC criterion is minimized over all possible values for z_0 .²

Following Cowell and Victoria-Feser (2008), we assume that the WMLE is an appropriate choice for estimating z_0 , but that a more efficient estimator is needed for estimating α . For this parameter, we use a member of the *optimal B-robust estimators* (OBRE) family (Hampel et al. (1986)), which is optimal in the sense of having the minimal asymptotic covariance matrix. Specifically, we use the standardized OBRE previously used, among others, by Victoria-Feser and Ronchetti (1994) in the context of parametric modelling of income distribution. In case of the power-law model (Equation 2), given a sample of n observations, the standardized OBRE is defined by

$$\sum_{i=1}^n \psi(z_i; \alpha) = \sum_{i=1}^n W_c(z_i; \alpha) \{s(z_i; \alpha) - a(\alpha)\} = 0, \quad (5)$$

where $s(z_i; \alpha) = 1/\alpha - \log(z) + \log(z_0)$ is the score function for the i^{th} observation. The weight function $W_c(z_i; \alpha)$ is

$$W_c(z_i; \alpha) = \min \left\{ 1, \frac{c}{\|A(\alpha)\{s(z_i; \alpha) - a(\alpha)\}\|} \right\}, \quad (6)$$

where $\|\cdot\|$ denotes the Euclidean norm. The matrix $A(\alpha)$ and vector $a(\alpha)$ ensure the asymptotic optimality and Fisher consistency of the estimator and are defined

²The estimation of k is equivalent to estimating z_0 , since k is the number of observations equal or greater to z_0 .

implicitly by

$$E[\psi(z; \alpha)\psi'(z; \alpha)] = [A(\alpha)'A(\alpha)]^{-1} \quad (7)$$

$$E[\psi(z; \alpha)] = 0. \quad (8)$$

Weights given in Equation 6 adjust observations, which are outliers for the assumed model. The constant c regulates the trade-off between efficiency and robustness – the lower c the more robust estimator but also the less efficient. For the power-law model, $c = 3$ gives OBRE with approximately 95% of asymptotic relative efficiency (ARE) in comparison with MLE, while $c=2$ has ARE of approximately 85%.³

4. Results

Table 1 summarizes the original data set on macroeconomic cumulative disasters (contractions of 10% or more over a short period) used in Barro and Jin (2011) and a richer data set used in Barro and Ursúa (2012).⁴ The latter adds observations for China, Egypt, Russia and Turkey (in case of GDP) and for Egypt, New Zealand, Russia and Turkey (in case of consumption, C). Sample sizes increase considerably with a newer data set by as much as 26 observations. Moreover, some of the new values are extremely large and therefore may have a significant impact on the tail behaviour of z . For example, the cumulative contraction in Russia over the period 1913-1921 was 71% in case of C and 62% in case of the GDP. In total, seven other observations are of size equal or greater to 40%.

Table 1. Descriptive statistics for macroeconomic disasters data

Data set	Variable	No. of obs.	Std. dev.	Mean	Max.
Barro-Jin sample					
	C	99	0.127	0.215	0.684
	GDP	157	0.126	0.204	0.736
Barro-Ursúa sample					
	C	125	0.133	0.217	0.707
	GDP	183	0.130	0.209	0.736

Table 2 shows the basic results of Barro and Jin (2011) (columns 2–3) and our replications (columns 4–7). Columns 4 and 7 provide the results of single two-parameter power-law robust fits with z_0 and α estimated using, respectively, the RC criterion and the OBRE. Columns 5–6 replicate the results of Barro and Jin (2011) with the newer data set using the MLE for single and double power-law model. Reassuringly, our estimates of α and γ are very close those obtained by Barro and Jin (2011) with their preferred double power-law model. In particular, the original paper estimated γ to be 3 in case of C and 2.75 in case of GDP. Our best robust estimate for the newer data set is the same for both macroeconomic variables and equal to 2.81 with a 95% confidence interval of (1.89, 4.27) for C and of (1.85, 3.72) for GDP.

³Detailed algorithms for computing OBRE can be found in Victoria-Feser and Ronchetti (1994) and Bellio (2007).

⁴Both data sets can be downloaded at <http://rbarro.com/data-sets/>.

The robust estimate of z_0 is noticeably different from the estimate of the double power-law cut-off δ only for the GDP data in the newer sample. However, even in this case the robust power-law exponent and the implied γ are remarkably close to those obtained with a double power-law model estimated with MLE.

Table 2. Three power-law fits and implied γ for macroeconomic disasters data

	Barro-Jin sample			Barro-Ursúa sample		
	Single fit	Double fit	Two-parameter single fit	Single fit	Double fit	Two-parameter single fit
<i>C data</i>						
α	6.27 (0.81) [4.96, 8.12]	4.16 (0.87) [2.66, 6.14]	4.12 (1.28) [2.48, 6.96]	6.11 (0.77) [4.86, 7.81]	3.72 (0.69) [2.77, 5.60]	3.43 (1.02) [2.29, 5.30]
β	— — —	10.10 (2.40) [7.37, 15.17]	— — —	— — —	10.14 (1.36) [8.39, 14.12]	— — —
δ	— — —	1.38 (0.13) [1.24, 1.77]	— — —	— — —	1.41 (0.08) [1.28, 1.54]	— — —
z_0	1.105	1.105	1.40	1.105	1.105	1.35
γ	3.97 (0.51) [3.13, 5.13]	3.00 (0.52) [2.16, 4.15]	2.96 (0.82) [1.93, 4.48]	3.99 (0.50) [3.17, 5.09]	2.82 (0.50) [2.31, 3.90]	2.81 (0.72) [1.89, 4.27]
<i>GDP data</i>						
α	6.86 (0.76) [5.56, 8.48]	3.52 (0.97) [2.39, 6.07]	3.40 (1.09) [2.19, 5.06]	6.56 (0.68) [5.45, 8.08]	3.50 (0.64) [2.53, 4.75]	3.57 (0.91) [2.20, 5.61]
β	— — —	10.57 (3.81) [8.67, 20.98]	— — —	— — —	10.29 (2.35) [8.08, 14.35]	— — —
δ	— — —	1.47 (0.15) [1.21, 1.69]	— — —	— — —	1.46 (0.10) [1.25, 1.64]	— — —
z_0	1.105	1.105	1.46	1.105	1.105	1.58
γ	4.33 (0.48) [3.50, 5.33]	2.75 (0.56) [2.04, 4.21]	2.74 (0.77) [1.85, 3.84]	4.27 (0.44) [3.54, 5.24]	2.75 (0.45) [2.30, 3.54]	2.81 (0.55) [1.85, 3.72]

Note: Standard errors are given in parentheses, 95% confidence intervals in square brackets. Single and double power-law fits for the Barro-Jin sample are taken from Barro and Jin (2011). The column for the two-parameter single power-law fit for the Barro-Jin sample and all columns for the Barro-Ursúa sample provide bootstrap standard errors and bias-corrected percentile bootstrap confidence intervals. γ is calculated using Barro and Jin (2011) assumptions about the values of the equity premium, σ and p (appropriately recalculated for the newer data set).

5. Conclusions

In this paper, we have successfully replicated the estimates of the coefficient of relative risk aversion obtained by Barro and Jin (2011) from a model linking equity premium puzzle to the power-law distribution of macroeconomic disasters. We have shown that their results are not sensitive to the addition of more data on extreme macroeconomic contractions and to the use of alternative methods taken from robust statistics and income distribution analysis.

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