University of Warsaw
Faculty of Economic Sciences

Working Papers
No. 4/2014 (121)

OLGA KIUILA
THOMAS F. RUTHERFORD

Economic modeling approaches: optimization versus equilibrium

Warsaw 2014

Economic modeling approaches: optimization versus equilibrium

Olga Kiuila
Faculty of Economic Sciences,
University of Warsaw
e-mail: kiuila@wne.uw.edu.pl

Thomas F. Rutherford
ETH Zurich
Centre for Energy Policy and Economics
e-mail: rutherford@aae.wisc.edu

Abstract

The paper discusses three general classes of nonlinear programming models. Our objective is to show how to represent optimization problem, such as nonlinear programs, in a compact format using extended mathematical programming. This is a useful tool, especially in cases when complementarity representation, such as mixed complementarity problems, could be difficult to apply. We reflect on the special features of the abstract neoclassical growth model with infinite horizon and illustrate four distinct approaches to computing equilibrium transition paths. The relationship between optimization and equilibrium modeling is explored. We conclude with a test of a new model representation (extended mathematical programming) using two common general equilibrium studies in the literature: Ramsey and Negishi. The new framework allows to define complementarity problems for a model formulated as an optimization problem, thus the resulting model becomes in fact a complementarity problem.

Keywords:
nonlinear programming, complementarity programming, extended mathematical programming, computable general equilibrium modeling, infinite horizon

JEL:
C02, C60, C61, C68, D58

1 Introduction

The finite-dimensional nonlinear optimization model can be written as:

$$\begin{aligned} \text{Given:} \quad & f : R^n \to R, \quad g : R^n \to R^m, \quad \ell, u \in R^n \\ \text{Find:} \quad & \max_{x \in R^n} f(x) \\ \text{s.t.} \quad & g(x) = 0, \quad \ell \le x \le u \end{aligned} \tag{1}$$

where $f(.)$ is the objective function, $g(.)$ describes the constraints, x is decision variable, ℓ and u represents lower and upper bounds respectively. The standard analytic technique for finding the solution to nonlinear optimization problem is the method of Lagrange multipliers, reducing the optimization to solution of a Karush-Kuhn-Tucker (`KKT`) system of inequalities. It can be solved numerically as a *nonlinear program* (`NLP`). The nonlinear program is a fundamental building block of economic equilibrium models, as it is used to describe the decision problems facing most of economic agents.

Another class of mathematical programs is complementarity model. It can be solved numerically as a *mixed complementarity problem* (`MCP`). This is particularly useful model when computing market equilibrium which cannot be represented by an optimization model. The `MCP` has no objective variable and is defined as:

$$\begin{aligned} \text{Given:} \quad & F : R^N \to R^N, \quad \ell, u \in R^N \\ \text{Find:} \quad & v, w, z \in R^N \\ \text{s.t.} \quad & F(z) = w - v, \quad w \ge 0, \ v \ge 0, \quad \ell \le z \le u \\ & w^T(z - \ell) = v^T(u - z) = 0 \end{aligned} \tag{2}$$

where $F(.)$ is the objective function, v and w are slack variables, z is decision variable. The `MCP` format involves the definition of complementarity pairs for variables and equations. This means that bounded variables must be mapped to complementarity inequality. By virtue of the `KKT` conditions, the complementarity problem class is large, including linear and nonlinear optimization problems as special cases. For cases in which the `MCP` corresponds to an optimization problem, the `MCP` represents the `KKT` system which has a particular skew-symmetric structure. The converse is not always the case: while all optimization problems can be expressed as complementarity problems, not all complementarity problems can be written down as optimization problems.

A new method, developed by Ferris et al. (2009), for representing complementarity problems through *extended mathematical programming* (`EMP`). This general method is diverse, including hierarchical models, bilevel optimization, disjunctive programs, and embedded complementarity problems. In this paper we focus on the last of these problem types which permits specification of a complementarity problem to be solved simultaneously with an optimization problem. We use General Algebraic Modeling System (GAMS) to describe this approach in details.

The idea of the EMP approach is to translate optimization problem into complementarity problem. An optimization model can be expressed as two linked models. These includes:

(i) an optimization problem defined in terms of decision variable, x, taking explicit multipliers, λ, and auxiliary variables, y, as given;

(ii) a set associated with auxiliary constraints which implicitly determine y.

The combined optimization model can be written as:

$$\begin{aligned} \text{Given:}\quad & f : R^{n+M+m} \to R, \quad g : R^{n+M+m} \to R^m, \quad H : R^{n+M+m} \to R^M \\ \text{Find:}\quad & \min_{x|y,\lambda} f(x; y, \lambda) \\ \text{s.t.}\quad & g(x; y, \lambda) \geq 0 \quad \perp \lambda \geq 0, \\ \text{where } y \text{ solves}\quad & H(x, y, \lambda) = 0 \end{aligned} \tag{3}$$

In this problem, $H(.)$ describes auxiliary constraints.[1] The NLP objective function $f(.)$ and constraints $g(.)$ are written with a semi-colon to separate the decision variable, x, from the remaining variables, y and λ, which are treated as parametric constants. Please note that y is treated as a parameter in the optimization problem, but it is a variable in the EMP formulation.

The EMP format requires to specify how to partition variables and equations in the model. That is, we need to indicate which variables are explicit multipliers[2] and which are auxiliary variables. Further, we need to indicate which equations are auxiliary constraints, i.e. equations that define $H(.)$ rather than $g(.)$. The complementarity problem corresponding to (3) has the form:

$$\begin{aligned} \nabla_x f(x; y, \lambda) - \lambda^T \nabla_x g(x; y, \lambda) = 0 \perp x \text{ free} \\ g(x; y, \lambda) \geq 0 \perp \lambda \geq 0 \\ H(x, y, \lambda) = 0 \perp y \text{ free} \end{aligned} \tag{3a}$$

This partitioning information is written in GAMS to an information file at execution time. The file indicates that $H(.)$ is associated with variable y. It would also indicate that variable λ represents the Lagrange multiplier associated with $g(x, y)$. The solver creates such mixed complementarity model (3a) consisting of the first order conditions for the nonlinear optimization problem together with the complementarity conditions which implicitly determine y.

These finite-dimensional programming formats (NLP, MCP and EMP) are useful tools for processing infinite horizon models. Our paper demonstrates a few ways in which these tools can be applied

[1] In the GAMS implementation of EMP, auxiliary variables and auxiliary equations are defined in pairs, so there are always exactly as many auxiliary variables as there are auxiliary constraints. The EMP example shown here assumes that all auxiliary constraints are equations and auxiliary variables are unconstrained. The auxiliary variable could alternatively been subject to upper and lower bounds, in which case the auxiliary constraint would be interpreted as a complementarity condition, i.e.:

$$H(x, y, \lambda) \gtrless 0 \quad \perp \ell \leq y \leq u$$

[2] These are dual (primal) variables in a primal (dual) optimization problem.

to the same model, producing identical results. We use a neoclassical dynamic model as an example of application for EMP, because the model is well studied in the economics literature. The rest of the paper is organised as following. Section 2 describes a logic of the Ramsey's growth model as an infinite horizon NLP. Next we define this model as an MCP and explain calibration issues. There are few characteristic features of the dynamic model which we discuss at greater length in section 4. In section 5, the EMP format is applied for this model and for one other CGE model (welfare model Negishi) in order to show how EMP works. Finally, we make a conclusion regarding the discussed model classes. The GAMS codes are available in the appendix. We indicate parameters and exogenous variables by lowercase letters to distinguish from the endogenous variables (uppercase letters).

2 Ramsey's Optimization Problem

One special type of complementarity problem (2) is the Arrow-Debreu computable general equilibrium (CGE) model. A general-equilibrium model consists of (i) profit-maximizing firms, (ii) markets, typically with supply and demand mediated through prices, and (iii) budget-constrained utility-maximizing households. Production activities in the Arrow-Debreu framework transform some goods and factors into others goods. These may include trade activities which transform domestic into foreign goods, activities which transform leisure into labor supply, and more conventional production activities which convert labor, capital and material inputs into products.

A CGE model we may represent as an optimization problem (1) and it should produce the same results as an equilibrium problem (2). To show this we will use a common model in economics - the neoclassical growth model. It is often presented as a Ramsey infinite-horizon dynamic optimization problem:

$$\begin{aligned} \max \quad & U(C_t) = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \frac{C_t^{1-\theta} - 1}{1-\theta} \\ \text{s.t.} \quad & C_t = f(K_t) - I_t, \quad K_{t+1} = (1-\delta)K_t + I_t, \quad K_0 = \bar{K}_0, \quad I_t \geq 0 \end{aligned} \tag{4}$$

where for simplicity it is assumed that there is no population growth. C_t, K_t, and I_t are aggregate consumption, capital, and investment in period t respectively, ρ is the time preference rate, δ is the rate of depreciation, and θ is an elasticity of the marginal utility of consumption (it is an inverse of the intertemporal elasticity of substitution $\sigma = 1/\theta$). It is typically assumed that the marginal product of capital is positive but decreasing, i.e. $f'(K) > 0$ and $f''(K) < 0$. The initial capital stock in period $t = 0$ is specified exogenously.

The maximand is a constant-elasticity-of-intertemporal substitution (CEIS) utility function. This additively separable utility function is simply a monotonic transformation of conventional CES utility function:

$$\hat{U}(C_t) = \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t C_t^{1-\theta} \right]^{1/1-\theta} \tag{4a}$$

The equivalence of $U(.)$ and $\hat{U}(.)$ is apparent when recalling that a monotonic transformation of utility does not alter the underlying preference ordering:

$$\hat{U} = \mathcal{V}(U) = [aU + \kappa]^{1/a}$$
$$\text{where } \kappa = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t = \frac{1+\rho}{\rho} \text{ and } a = 1-\theta$$

$\mathcal{V}(.)$ is a monotonic transformation ($\mathcal{V}' > 0$), hence optimization of $U(.)$ and $\hat{U}(.)$ yield identical demand functions:

$$C_t = \frac{M}{E^{1-\sigma}} \left(\frac{\rho}{(1+\rho)^{t+1} P_t} \right)^{\sigma} \tag{5}$$

$$\text{where } E(p) = \left(\frac{\rho}{1+\rho} \right)^{\sigma/(1-\sigma)} \left(\sum_t \left(\frac{1}{1+\rho} \right)^{t\sigma} P_t^{1-\sigma} \right)^{1/(1-\sigma)} \tag{5a}$$

where M is income, P_t is market price, and $E(.)$ is expenditure function. Alternatively, recall that preference orderings are defined by the marginal rate of substitution. In both of these models (4) and (4a) we have:

$$\frac{\partial U / \partial C_{t+1}}{\partial U / \partial C_t} = \frac{\partial \hat{U} / \partial C_{t+1}}{\partial \hat{U} / \partial C_t} = \frac{1}{1+\rho} \left(\frac{C_t}{C_{t+1}} \right)^{1-\sigma} \tag{6}$$

Among advantages associated with the use of linearly homogeneous representation of preferences (4a), it is simplifies reporting of welfare changes. Hicksian-equivalent variations is trivially measured with `CES`: a 1% change in $\hat{U}(.)$ corresponds to a 1% equivalent variation in income. Thus we will relate the Ramsey intertemporal preferences to `CES` utility functions which are commonly employed in static general equilibrium models.

3 Ramsey's Equilibrium Problem

Equilibrium in the Arrow-Debreu model is characterized by three classes of equations mentioned in Section 2. To be able to formulate those equations in terms of `MCP` modelling we have to define also complementarity pairs. To illustrate it, we implement a specific policy experiment with Ramsey model, letting τ_t represent the capital income tax applied in year t.

Unlike the formulation (4) and many macroeconomics textbooks that express aggregate output Y_t as a function of capital stock alone: $Y_t = f(K_t)$, we will use two production factors. In our representation of the Ramsey model, it is convenient to work with a constant-returns production function in which we have inputs of both labor and capital:

$$Y_t = F(K_t, \bar{L}_t) \tag{7}$$

When labor is in fixed supply ($\bar{L}_t$), the production function exhibits diminishing returns to capital. There is therefore no loss of generality by formulating the model on the basis of a constant returns to scale technology.

(i) Market clearance conditions and associated market prices[3]

- Output market: $Y_t = C_t(p_t, M) + I_t \quad \perp p_t \geq 0$
- Labor market: $\bar{L}_t = a_L(r_t^K(1+\tau_t), p_t^L)\ Y_t \quad \perp p_t^L \geq 0$
- Market for capital services: $K_t = a_K(r_t^K(1+\tau_t), p_t^L)\ Y_t \quad \perp r_t^K(1+\tau_t) \geq 0$
- Capital stock: $K_{t+1} = (1-\delta)K_t + I_t \quad \perp p_t^K \geq 0$

where p_t, p_t^L, r_t^K, and p_t^K represents market price, wage rate, capital rental rate, and capital purchase price respectively. The compensated demand functions for labor (a_L) and capital (a_K) are the partial derivatives of the unit cost function.

(ii) Zero profit conditions and associated activities

- Production process: $p_t = c(p_t^L, r_t^K(1+\tau_t)) \quad \perp Y_t \geq 0$
- Investment: $p_t \geq p_{t+1}^K \quad \perp I_t \geq 0$
- Capital stock: $p_t^K = r_t^K + (1-\delta)p_{t+1}^K \quad \perp K_t \geq 0$

where $c(.)$ is a price index for value-added. The only activity level which could possibly fall to zero would be investment, and that would only happen in a policy scenario which resulted in a substantial reduction in the return to capital.

(iii) Income balance

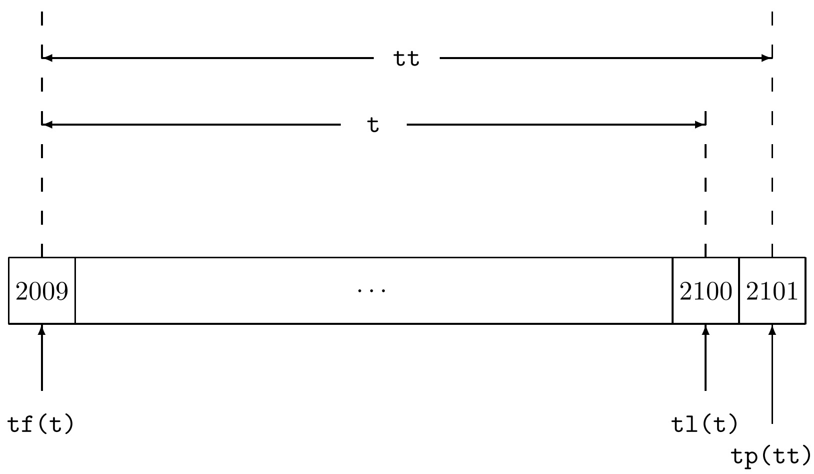
- $M = p_0^K \bar{K}_0 + \sum_{t=0}^{\infty} \left(p_t^L \bar{L}_t + \tau_t r_t^K K_t\right) - p_{T+1}^K K_{T+1}$

The choice of T depends on the nature of the policy experiment and the speed with which the model converges to a new steady-state growth path. A benchmark steady-state can be reached through a calibration technique. The idea is that if a model is parameterized on the basis of economic transactions in a reference equilibrium with policy parameters $\bar{\tau}$, then the model should be able to reproduce that equilibrium when given policy parameter $\tau = \bar{\tau}$. Such consistency check is required for any `CGE` modeling.

[3]The demand functions employed in this model assure that all prices will be nonzero in equilibrium. There is no formal need, therefore, to associated prices with market clearance conditions, as would be required in a conventional complementarity problem. We provide an associated here in order to help understand how the model might be extended with demand functions which would admit zero prices.

4 Calibration technique for infinite horizon

Benchmark replication is a common strategy for evaluating logical consistency of both static and dynamic general equilibrium models. An immediately evident challenge which Ramsey's model presents for economists commonly working with static models is the infinite time horizon. We are only able to process models with a finite number of variables. When moving from static to dynamic models, we move from computing the exact equilibrium to approximating the infinite-horizon transition path using tools for solving finite dimensional optimization and complementarity problems.



`tt`	Time horizon (with the first year of the post-terminal period)
`t(tt)`	Time period over the model horizon
`tf(tt)`	First period of the model (2009)
`tl(tt)`	Last endogenous period (2100)
`tp(tt)`	Post-terminal period (the element of set `tt` which is not an element of `t`, 2101)

Figure 1: Sets Defining Time Structure of the Ramsey model

How an infinite horizon model can be effectively represented by a T-period approximation has been explained by Lau et al. (2002). The solution process requires to decompose the consumer's problem: a one run through time period T and another for the post-terminal period $(T + t)$. The Figure 1 shows a relationship between time periods over the model horizon.

If a growth model is calibrated to a steady-state growth path, this can serve the same role as the benchmark equilibrium dataset in a static model. One problem with this strategy is that in many cases, the base year data may be far removed from a steady-state growth path. In this situation, as a case, it may be possible to adjust the database (only for the purpose of debugging the model formulation) such that some versions of the model are consistence with the steady-state growth.

It is customary, however, first to solve a business-as-usual growth path as a reference case prior to solving other cases, so policy analysis consists of comparing one adjustment path with another.

Here are the steps involved in sorting out the stead-state conditions with related investments and capital earnings in a static data set which is consistent with a steady-state growth path.

(i) Imputed Capital Cost:

The simplest approach is to set up the model along a steady-state growth path in which interest rate ($\bar{r}$) and growth rate ($\bar{g}$) are given and assumed constant. The zero-profit condition for investment reveals the price level for capital (p_t^K):

$$p_{t+1}^K = \frac{p_t^K}{1+\bar{r}} = P_t \tag{8}$$

The base year price of capital, assuming the Harberger convention (Harberger, 1962), is then: $\bar{p}_0^K = 1 + \bar{r}$.

(ii) Marginal Product of Capital:

The zero profit condition for capital determines the capital rental price (r_t^K):

$$p_t^K = r_t^K + (1-\delta)p_{t+1}^K \tag{8a}$$

Note that in a static modelling we do not have p_t^K, but only r_t^K. Substituting the values of p_t^K and p_{t+1}^K reveals that the base year rental price of capital is sufficient to cover interest plus depreciation: $\bar{r}_0^K = \bar{r} + \delta$.

(iii) Macro Reconciliation:

Finally, consider the market clearance condition for capital in the first period:

$$K_1 = \bar{K}_0(1-\delta) + \bar{I}_0 = (1+g)\bar{K}_0 \tag{8b}$$

This implies that base year investment can be calculated on the basis of growth and depreciation of the base year capital stock: $\bar{I}_0 = \bar{K}_0(g+\delta)$. We then can use $\bar{r}_0^K$ to determine $\bar{K}_0$ on the basis of the value of capital earnings in the base year ($\bar{V}_0^K$):

$$\bar{I}_0 = \bar{V}_0^K \, \frac{g+\delta}{r+\delta} \tag{8c}$$

The problem that arises in applied models is that $\bar{I}_0$ and $\bar{V}_0^K = \bar{K}_0\bar{r}_0^K$ will not satisfy this relation for arbitrary values of $\bar{g}$, $\bar{r}$ and δ. Part of the art of economic equilibrium analysis involves reconciliation of theory and data. It is perhaps not surprising that when we add more theory (e.g., intertemporal decisions) we introduce more constraints on the underlying database.

The terminal approximation in a model with a single consumer is quite simple, because we do not need to define a value of terminal assets. A precise approximation only requires that the terminal capital stock closely follows the steady-state value. This is consistent with the convergence in growth rates of all macroeconomic variables, the most sensitive of which is investment. Post-terminal capital stock, K_{T+1}, can be chosen such that the terminal growth rate of capital matches aggregate economic growth.

In models with multiple consumers living beyond period T, it would be necessary to account for which of these agents owns the assets. Some agents may have negative asset positions at the end of the model (particularly in overlapping generations models where young households accumulate debt which is repaid in middle age).

5 Optimization versus Complementarity

The NLP models have lower precision compared to alternative models formulated with `MCP`. For example, Lau et al. (2002) shows that Ramsey model formulated with `NLP` substantially underestimate the level of debt. Complementarity determines what activities run, who produces what, and who consumes what. The cost of this precision is a requirement explicitly represent both primal and dual optimality conditions. Optimization approach is more compact and it assumes that we control the complete system. In order to use the advantages of both approaches, we apply a new tool `EMP` discussed in section 1.

5.1 Extended Mathematical Programming

Extended Mathematical Programming allows to define a model in a compact way (`NLP`) and use a precision of complementarity approach (`MCP`). Adjusting optimization problem (4) to a policy experiment with capital income tax described in section 3, the Ramsey optimization problem (3) looks as following:

$$\max_{C_t, I_t, K_t | K_{T+1}, r_t^K} \left(\frac{1}{1+\rho}\right)^t \frac{C_t^{1-\theta} - 1}{1-\theta} - \sum_t \tau_t r_t^K K_t \tag{9}$$

$$\text{s.t.} \quad \psi l_t^{1-\alpha} k_t^{\alpha} = Y_t = I_t + C_t, \quad K_{t+1} = (1-\delta)K_t + I_t, \quad k_t = K_t \ \perp r_t^K, \quad l_t = \bar{L}_t$$

where the auxiliary variable K_{T+1} and the associated auxiliary equation $I_T / I_{T-1} = 1 + g$ implements the steady-state condition for terminal investment growth.[4] The model is initially calibrated to a steady-state growth path with no capital income tax ($\tau_t = 0$). Later we evaluate the economic consequences of such taxation assuming lump-sum revenue recycling to the representative agent.

The EMP formulation (optimization problem) can be defined in primal or dual form. In economic analysis, preferences may be described by a primal form utility function, $U(C)$, or, equivalently,

[4]This termination technique is hence characterized as *state variable targeting*. A state variable's terminal value is chosen such that the growth rate of the associated control variable is consistent with steady-state growth.

using a dual form indirect utility function, $V(p, M) = M/e(p)$. When preferences are homothetic, the unit expenditure function (5a) conveys all of the information concerning the underlying preferences. Closed form expressions for $U(.)$ and $V(.)$ are not always available. The most flexible functional forms have a dual form but no explicit primal form. The constant elasticity functions used in the present analysis admit both primal and dual forms. Thus the Ramsey maximization problem (9) can be represented using the expenditure function in place of the utility function:

$$\min_{p,p^L,p^K,r^K|I,K} \sum_t p_t^L L_t + p_0^K K_0 - p_{T+1}^K K_{T+1} + \tau_t r_t^K K_t - \bar{M} e(p) \tag{10}$$
$$\text{s.t.} \quad (p_t^L)^{1-\alpha}(r_t^K(1+\tau_t))^\alpha = p_t, \quad p_t^K = p_{t+1}^K(1-\delta) + r_t^K \;\perp K_t, \quad p_t \geq p_{t+1}^K \;\perp I_t$$

The constraints for the dual optimization problem are the zero profit conditions for the underlying equilibrium problem. The income balance is included into the objective function (10). The market clearance conditions for the equilibrium are incorporated in the first-order optimality conditions. According to Sheppard's lemma, the demand in year t equals the utility level times the gradient of the expenditure function: $C_t(p, M) = V(p, M)\partial e(p)/\partial p_t$. Substituting into the first order condition, we see that the optimality conditions for the dual correspond to the market clearance conditions which appears explicitly in the primal.

5.2 EMP and MCP with GAMS

Primal and dual forms with EMP require to distinguish between primal decision variables and dual variables. Complementary pairs should be defined only for dual variables in the primal EMP formulation and for primal variables in the dual EMP formulation. Also auxiliary constraints requires directly to determine a complementary variable. All other variables do not need complementary pairs with EMP, but primal (dual) multipliers should be assigned to equations in dual (primal) forms. On the other hand, the MCP formulation (equilibrium problem) requires complementary pairs for all variables.

In the case of Ramsey equilibrium problem, there are six primal variables $Y_t, I_t, K_t, C_t, LD_t, KD_t$, four dual variables p_t, r_t^K, p_t^K, p_t^L, income variable, and auxiliary variable. The GAMS code is available in Appendix A1.[5] The same model formulated as a primal EMP model (Apeendix A2) has only one dual varible (capital rental price r_t^K), seven primal variables, and decision variable. The additional to MCP primal variables is utility level.

The EMP framework is available with GAMS software. This framework requires to declare complementary pairs in a separate file. This information file redefines NLP formulation into MCP. This means that the information file for Ramsey model should contain a declaration for one implicit auxiliary variable-equation pair and one explicit dual variable-equation pair. The explicit

[5] We use a GAMS tool \$*macro* that allows to formulate some equations with no declaration of complementarity pairs. These equations (cost function, compensated factors demand, consumption function) just help to formulate necessary equilibrium conditions described in Section 3.

dual variable, r_t^K, is introduced in order to provide an explicit measurement of capital earnings. The associated equation for this dual variable is a market clearing condition for capital service. The implicit auxiliary variable is a post-terminal capital stock K_{T+1}. The associated auxiliary equation is explicitly defined.

Next, an option file should be created in order to save results from reformulation of NLP framework into MCP. Finally, we verify calibration using zero iterations. If the baseline growth path is replicated, then we able to simulate scenarios. Our experiment is based on the scenario, where capital income tax $\tau_t = 25\%$ is applied in year $t = 6$. For this purpose the information file should be redefined.

Thus in a primal EMP formulation only market clearing conditions are applied explicitly, while MCP format requires to define all necessary conditions. Instead of zero profit condition and income balance we rather have to define production and utility functions with primal EMP.[6] In order to get a positive Lagrange multiplier, the objective function includes a utility function with a negative sign. This means that we have to minimize the objective function in the primal EMP model in order to get the same results as in the MCP model. Since our simulation experiment includes a lump-sum recycling of capital income tax revenue, the objective function includes also that revenue.

Alternatively, the dual EMP formulation requires to declare only zero-profit condition (Appendix A3). Market clearing condition is replaced by expenditure function and income balance is replaced by objective function. Here we have five dual variables, two primal variables, and decision variable. The information file includes complementary pairs only for auxiliary equation and for primal variables. Since auxiliary constraint refers to investment in both periods I_T and I_{T-1}, then we have to declare a separate complementary pairs for investments in both periods.

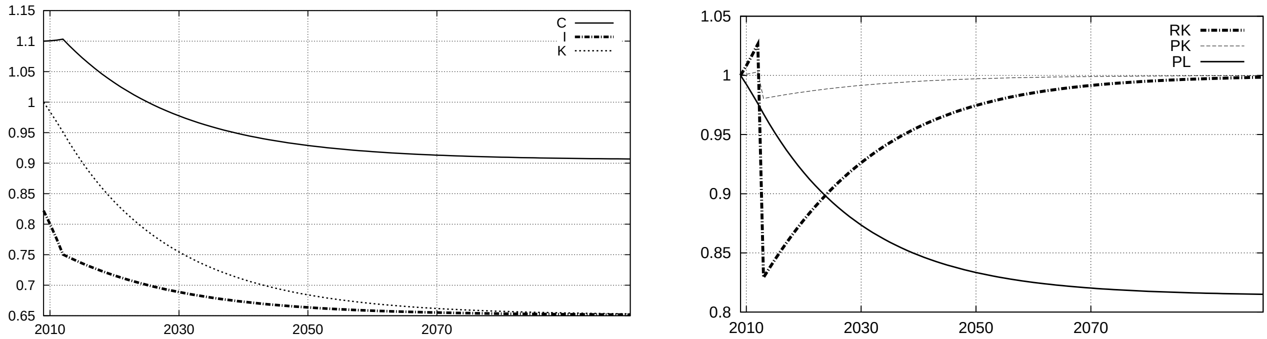
5.3 Application of extended mathematical programming

5.3.1 Ramsey model with tax on capital earnings

We have assigned a permanent tax on capital earnings ($RK_t * K_t$) of 25% beginning in 2013 using a model described in Sections 3 and 5.1. Cobb-Douglass production structure allows to keep the perfect foresight model simple. The equilibrium is computed assuming that economic agents anticipate the application of the tax, resulting in a sharp response in investment and other economic variables to the new economic environment. Exactly the same results were obtained by MCP and both EMP versions.

Selected results are presented in Figure 2. When economic agents' behavior depend on forward-looking expectations, then a credible announcement of a future change in policy has an effect already before the new policy is implemented. Immediately after 2013, investment will be

[6] No *$macro* tool can be used to define these functions.



a. Indices of consumption (C), investment (I) and capital stock (K)

b. Capital rental rate (RK), capital purchase rate (PK) and wage rate (PL)

Figure 2: Simulation of tax on capital in year 5 (Ramsey MCP Model)

decreasing because saving is less than required to sustain a constant capital stock and consumption will be slightly increasing. At time 2013 (year 5) the economy reaches a new saddle path, the high taxation on capital income begins, and the rate of return becomes considerably lower. Future consumption becomes relatively expensive and elasticity of intertemporal substitution cause a shift in consumption from the future to the present. The economy gradually approaches to the new steady state.

The Walras law implies that capital flows may be positive in some periods or negative in others, but the sum of their present values is zero over the planning horizon (Manne and Rutherford, 1994). Over time the tax leads to a smooth reduction in the steady-state capital stock, while capital purchase price returns to steady state after 44 years. Investment is typically the most sensitive item, it decrease by over 30% in 2013 and then decrease smoothly, while capital rental price returns to a steady state after 69 years. Tax on capital earnings leads to smooth decrease of labor price, because labor supply has exogenous growth path, while capital is adjusted endogenously.

5.3.2 Negishi model with sales tax

Negishi (1960) looked for the set of Pareto efficient allocations (as solutions of a central planner's problem) to show (using Second Welfare Theorem) that there exists an allocation which constitutes a general equilibrium. Solving a planner's problem is easier than directly obtaining computations of competitive equilibria. In doing so we have to be careful in applying optimal control theory because there are exists difficulties with respect to differentiability and boundness of equilibrium allocations (van Geldrop and Withagen, 1994). However, when the number of commodities is large relative to the number of consumers, then the excess demand approach (Scarf, 1973) is inefficient compared to the Negishi approach.[7]

[7] Ginsburgh and Keyzer (2002, p.113) provides the following explanation for such inefficiency. Every agent in the

If the economy is Walrasian, then all equilibrium allocations are Pareto efficient. Otherwise, the parametrization used to characterize equilibrium allocations will fail to capture inefficient equilibria. For examples see Tirelli (2010). A competitive equilibrium can be also represented through a welfare optimum with nonzero welfare weights, which satisfy budget constraints for all households. Negishi's approach assumes arbitrary weights for consumers. Ginsburgh and van der Heyden (1988) shows how to represent market imperfections (such as taxes or maximum prices) in the Negishi format. Such allocation can be found by maximizing a weighted sum of the welfare functions and government operations subject to the technological and system constraints:

$$\max_{C,X|\theta,p} \sum_h \theta_h \ln U_h(C_{g,h}) + \sum_g t_g p_g Y_g(X_{f,g})$$
$$\text{s.t.} \quad \sum_h C_{g,h} \leq Y_g(X_{f,g}), \quad \sum_g X_{f,g} \leq \sum_h \bar{e}_{h,f} \tag{11}$$

where welfare weights $\theta_h = \sum_f e_{h,f} / \sum_h \sum_f e_{h,f}$ are interpreted as Negishi coefficients (weights). They depend on fixed factor endowment $e_{h,f}$ and they hold the budget constraint for every household h. Here sales tax t_g is proportional to output value, while output price p_g is a vector of Lagrange multipliers associated with market clearing condition for goods g.

There are three ways to solve this model with taxes. If we use NLP method, we have to apply a sequential joint maximization approach (Rutherford, 1999) that (i) solves the NLP with no tax using the fixed prices and weights for primal NLP or fixed income and weights for dual NLP.[8] (ii) updates the prices based on the new marginal of market clearing conditions for both factors and goods from the NLP solution with tax; update the weights based on equilibrium income (iii) computes the difference between old and new weights until the difference is small enough. As the weights converge, the agents will move toward balanced budgets, where their incomes equal to their expenditures. This means that no income balance can be formulated with a primal NLP model because no endogenous price of factors in a primal NLP. Instead, we use a loop with two additional variables: factor price w_f and equilibrium income $M_h = \sum_f w_f e_{h,f} + 1/card_h \sum_g t_g p_g Y_g$.

Let's consider a 2x2x2 model in which a set of consumers $(H1, H2)$ are each endowed with a fixed quantity of factors $(F1, F2)$. The first household posses only $F1$ and the second household - $F2$. The solution consists of a consumption vector for each agent and a set of prices for each good $(G1, G2)$ such that each agent maximizes her utility at this consumption level. Utility is given by a CES function and production technology is described by Cobb-Douglass function. In order to initialize the weights, they were chosen randomly (see Appendix 4). The scenario with a sales tax on first good $t(G1) = 0.5$ implies that income of the first household goes up because the price of $F1$ decrease less than $F2$. Production of $G1$ requires relatively less $F1$, thus a sales tax implies a

excess demand format (fixed point approach) solves his own welfare function and the feedback is an adjustment on prices only. While in the Negishi approach, there is a single welfare function and the feedback is an adjustment on prices and weights.

[8] Weights and prices should be fixed with primal NLP because these variables define market clearing conditions for factors and goods respectively. Weights and supply should be fixed with dual NLP because these variables define zero profit conditions for factors and goods producers respectively.

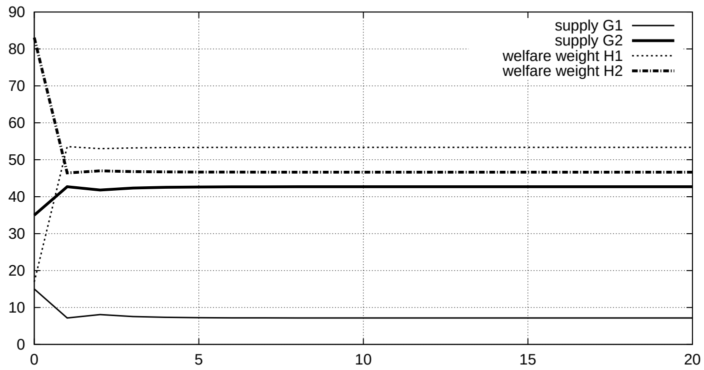


Figure 3: Simulation of sales tax on good G1 (Negishi NLP Model)

relatively less decrease in price of $F1$. As a result, the Negishi coefficient for $H1$ is higher than $H2$(Figure 3).

Welfare adjustment process enables the computation of an equilibrium, adjusting the weights assigned to each individual in the social welfare function. We can confirm this result using a second method to solve this model - EMP. It is allow to find the Negishi coefficients directly. The objective function and primal constraints are the same as NLP model, but side constraints for welfare coefficients and income should be also included directly into the main body of the model. Then for dual variables and side constraints we have to assign complementary pairs: w_f for factors market clearing condition and p_g for goods market clearing condition, income share for θ_h, and income definition for M_h.

Both codes for primal NLP and EMP Negishi model are available in Appendix 4. The dual representation is alike. NLP approach requires few iterations (Figure 3) to get the same solution as EMP. Solution speed is five times greater with EMP than NLP.

6 Conclusion

Equilibrium problem described by MCP can be interpreted as first-order conditions for optimization problem described by NLP. The disadvantage of the MCP approach is a model specification, because complementarity requires explicit representation of both optimality conditions (primal and dual). It might be difficult to design such structure for a model. The example is the DICE model. In this case the EMP framework helps. This paper outlines that approach within GAMS modeling system for an analysis of interactions among forward-looking players in dynamic environment (Ramsey model).

Optimization approach assumes that we control the complete system and the model representation is compact. Three classes of the equilibrium conditions (market clearing, zero profit, and income balance) required in the complementarity framework are reduced just to one class in the optimization framework (market clearing for primal approach or zero profit for dual approach). Instead of zero-profit conditions and income balance, we have to formulate production function, utility function, and objective function in the primal optimization approach. Instead of market-clearing conditions and income balance, we have to formulate cost function, expenditure function, and objective function in the dual optimization approach.

Mathematical programs with optimization problems in their constraints have a long history in operations research. Yet Ferris and Sinapiromsaran (2000) found that forming NLP model and solving this as a complementarity problem is a viable approach. With the extended mathematical programming it works well, according to our tests for three economic models. EMP allows for computational advances, such as automatic differentiation technique, to eliminate errors that occur without automation and to conduct models with formulations otherwise too time-consuming to consider. Specifically:
1) NLP formulation is compact compared to MCP. Thus EMP allows to use advantage of NLP to *formulate* a model.
2) NLP solvers uses quite high memory and superbasic variables. MCP does not require superbasics. Thus EMP allows to solve better *large* models.
3) NLP framework requires that the model horizon is sufficiently long to converge to the steady-state. MCP framework better approximate the infinite horizon and it is virtually insensitive to the model horizon (Boehringer et al., 2007). Thus EMP allows to improve model *robustness*.
4) we have to be careful in applying optimal control theory because there are exists difficulties with respect to differentiability and boundness of equilibrium allocations. Thus it better to apply the First Welfare Theorem (equilibrium approach) rather than the Second one (optimization).
5) MCP format accommodates a wider range of economic complexities, than NLP (more goods, more regions, tax revenue recycling, etc.)
6) MCP relaxes the integrability constraints (the shadow prices coincide with market prices) imposed by the NLP framework. Thus EMP allows to accommodate second-best settings that reflect *initial inefficiencies*.

Every NLP model can be represented with MCP framework but not vice versa. Complementarity slackness conditions determine which prices and activity levels are at zero level, and which are positive. We use a bridge (EMP framework) that allows to formulate a model as an optimization problem, but solve it as a complementarity problem. In the Ramsey experiment we have verified whether MCP and EMP produce the same results using a simulation of capital income tax. In the Negishi model we have analysed whether NLP and EMP produce the same results using a simulation of sales tax.

References

Boehringer, C., Loeschel, A., Rutherford, T., 2007. Decomposing the integrated assessment of climate change. Journal of Economic Dynamics and Control 31, 683–702.

Ferris, M., Sinapiromsaran, K., 2000. Formulating and solving nonlinear programs as mixed complementarity problems, in: Nguyen, V., Strodiot, J., Tossings, P. (Eds.), Optimization. Springer Berlin Heidelberg. volume 481 of *Lecture Notes in Economics and Mathematical Systems*, pp. 132–148.

Ferris, M.C., Dirkse, S.P., Jagla, J.H., Meeraus, A., 2009. An extended mathematical programming framework. Computers and Chemical Engineering 33, 1973–1982.

van Geldrop, J., Withagen, C., 1994. General equilibrium in an economy with exhaustible resources and an unbounded horizon. Journal of Economic Dynamics and Control 18, 1011–1037.

Ginsburgh, V., van der Heyden, L., 1988. On extending the Negishi approach to computing equilibria: The case of government price support policies. Journal of Economic Theory 44, 168–178.

Ginsburgh, V., Keyzer, M., 2002. The Structure of Applied and General Equilibrium Models. MIT Press.

Harberger, A., 1962. The incidence of the corporation income tax. Journal of Political Economy 70, 215–240.

Lau, M., Pahlke, A., Rutherford, T., 2002. Approximating infinite-horizon models in a complementarity format: A primer in dynamic general equilibrium analysis. Journal of Economics Dynamics Control 26, 577–609.

Manne, A., Rutherford, T., 1994. International trade, capital flows and sectoral analysis: Formulation and solution of intertemporal equilibrium models, in: Cooper, W., Whinston, A. (Eds.), New Directions in Computational Economics. Springer Netherlands. volume 4 of *Advances in Computational Economics*, pp. 191–205.

Negishi, T., 1960. Welfare economics and existence of an equilibrium for a competitive economy. Metroeconomica 12, 92–97.

Nordhaus, W., 1994. Managing the global commons. MIT Press.

Nordhaus, W., 2008. A Question of Balance. Yale University Press.

Rutherford, T., 1999. Sequential joint maximization, in: Weyant, J. (Ed.), Energy and Environmental Policy Modeling. Kluwer. volume 18 of *International Series in Operations Research & Management Science*, pp. 139–175.

Scarf, H., 1973. The computation of Economic Equilibria. Yale Univ. Press.

Tirelli, M., 2010. Negishi's approach to Walrasian equilibria. Handout 2. Department of Economics, Roma Tre University.

Appendix A: GAMS Code

A.1: The MCP approach for Ramsey Model

```
$TITLE Ramsey Model -- MCP formulation calibrated to base year data

$include dataCES

Nonnegative Variables

*       Activity levels:
        Y(t)    Output
        I       Investment
        K       Capital stock

*       Prices:
        P       Output price
        RK      Rent of capital
        PK(tt)  Capital price
        PL(t)   Wage rate

*       Income level:
        RA      Representative agent

*       Auxiliary variable (targeting of steady-state):
        TK      Post-terminal capital stock


*       Cobb-Douglas Price index for value-added:

$macro PKL(t)  (PL(t)**(1-kvs) * {RK(t)*(1+taxk(t))}**kvs)

*       Cobb-Douglas Compensated labor demand:

$macro LD(t)   (l0   * PKL(t)/PL(t))

*       Cobb-Douglas Compensated capital demand:

$macro KD(t)   (kd0  * PKL(t)/{RK(t)*(1+taxk(t))})

*       CEIS Consumption demand:

alias (t,t_);
$macro expend  (sum(t_, thetac(t_) * (P(t_)/pref(t_))**(1-sigma))**(1/(1-sigma)))
$macro CD(t)   (qref(t)*c0 * (RA/(expend*ra0)) * (expend*pref(t)/P(t))**sigma)

Equations

* Market clearing condition
         markety(t)         goods market,
         marketrk           capital market
         marketl(t)         labor   market
```

```
        marketk            investment market

* Zero profit condition
        profity(t)         production,
        profiti(tt)        investments
        profitk            capital

* Income balance
        income            representative agent budget

* Auxiliary variables:
        tkdef             post-terminal capital stock;

markety(t)..         Y(t)*y0 =e= I(t)*i0 + CD(t);
profity(t)..         PKL(t)*(l0 + kd0) =e= P(t)*y0;
marketl(t)..         qref(t)*l0 =e= Y(t)*LD(t);
marketk(tt)..        K(tt-1)*k0*(1-delta) + I(tt-1)*i0 + k0$tf(tt) =e= K(tt)*k0$t(tt) + TK$tp(tt);
marketrk(t)..        K(t)*kd0 =e= Y(t)*KD(t);
profitk(tt)$t(tt)..  PK(tt)*k0 =e= PK(tt+1)*k0*(1-delta) + RK(tt)*kd0;
profiti(tt)$t(tt)..  P(tt) *i0 =e= PK(tt+1)*i0;
tkdef..              sum(tl(t),  I(t)/I(t-1) - Y(t)/Y(t-1)) =e= 0;
income..             RA =e= sum(t, PL(t)*qref(t)*l0)
                          + sum(tf, PK(tf)*k0)
                          - sum(tp, PK(tp)*TK)
                          + sum(t,  taxk(t)*RK(t)*KD(t)*Y(t)) ;


model ramseym /markety.P, profity.Y, marketl.PL, marketk.PK, marketrk.RK, profitk.K, profiti.I,
               tkdef.TK, income.RA/;

*       Assign steady-state equilibrium values for quantities
*       and prices:

Y.L(t) = qref(t);
I.L(t) = qref(t);
K.L(t) = qref(t);
P.L(t)  = pref(t);
RK.L(t) = pref(t);
PL.L(t) = pref(t);

* The steady-state price of capital is the output price
* times one plus the interest rate:

PK.L(tt) = (1+r) * pref(tt);
TK.L = k0 * (1+g)**card(t);
RA.FX = ra0;

ramseym.iterlim = 0;
solve ramseym using mcp;
ABORT$(ramseym.objval > 0.001) "Model does not calibrate.";

* Apply a tax on capital inputs of 25% beginning in year 6:

taxk(t)$(t.val > 2012) = 0.25;
```

```
ramseym.iterlim = 100000;
solve ramseym using mcp;

*       Extract consumption as a variable for use in the report:

variable        C(t)    Consumption;

$ondotl
C.L(t) = CD(t);

$include report
```

A.2: The Primal EMP approach for Ramsey Model

```
$TITLE Ramsey Model -- EMP primal formulation calibrated to base year data

$include dataCES

parameter       psi(t)  Production function scale factor
                rho     Primal form elasticity;

psi(t)     = y0 /(l0**(1-kvs) * kd0**kvs);
rho = 1 - 1/sigma;

Variable      OBJ     Objective function;

Nonnegative Variables

* Primal decision variables:
        Y(t)  Output
        I     Investment
        K     Capital stock
        C     Consumption
        KD    Capital demand
        LD    Labor demand
        W       utility

* Dual (auxiliary) variables used in the EMP framework:
        RK(tt)  Rental price;

Equations

* Market clearing condition:
         markety(t)         goods market,
         marketrk           capital market
         marketl(t)         labor   market
         marketk            investment market

* Functions:
         production(t)      production function
         objdef             objective function
         welfare            utility function

* Auxiliary constraint:
         tkdef              post-terminal capital stock;

objdef..             OBJ =e= -W + sum(t,  taxk(t)*RK(t)*KD(t));
welfare..            ra0 * sum(t, thetac(t)*(C(t)/(qref(t)*c0))**rho)**(1/rho) =e= W;
production(t)..      psi(t) * LD(t)**(1-kvs) * KD(t)**kvs =e= Y(t);
markety(t)..         Y(t) =e= I(t) + C(t);
marketl(t)..         qref(t)*l0 =e= LD(t);
marketrk(t)..        K(t)*rk0 =e= KD(t);
marketk(tt)..        K(tt-1)*(1-delta) + I(tt-1)+ k0$tf(tt) =e= K(tt);
tkdef(tl(t))..       I(t)/I(t-1) =e= Y(t)/Y(t-1);
```

```
*       Declare the model with all of the equations defined here:

model ramsey /all/;

*       Assign steady-state values:
C.L(t) = qref(t)*c0;
Y.L(t) = qref(t)*y0;
I.L(t) = qref(t)*i0;
K.L(tt)= qref(tt)*k0;
KD.L(t)= qref(t)*kd0;
LD.L(t)= qref(t)*l0;
RK.L(t)= pref(t);
W.L   = ra0 ;
OBJ.L = -ra0;

*       Assign the dual multipliers consistent with this growth path:
production.m(t) = pref(t);
markety.m(t)    = pref(t);
marketrk.m(t)   = pref(t);
marketl.m(t)    = pref(t);
marketk.m(tt)   = pref(tt)*(1+r);
welfare.m       = 1;

*       Write an information file which indicates which auxiliary
*       variables and constraints:

file empinfo / '%emp.info%' /;
put empinfo;
loop(t$taxk(t),         put /  'dualvar ' RK.tn(t)     ' ' marketrk.tn(t));
loop((tl,tp),           put /  'dualequ ' tkdef.tn(tl) ' ' K.tn(tp));
putclose;

*       Generate an EMP option file requesting that the scalar
*       MCP file and a dictionary both are saved:
$onecho >emp.opt
filename=emp-p-scalar.gms
Dict=emp-p-scalar.txt
$offecho
ramsey.optfile = 1;

*       First solve has no iterations to replicate the baseline growth path:
ramsey.iterlim = 0;
SOLVE ramsey USING emp minimizing OBJ;

*       Apply a tax on capital inputs of 25% beginning in 2013:
taxk(t)$(t.val > 2012) = 0.25;

*       Regenerate the EMP control file which changes when taxk is modified:
put empinfo;
loop(t$taxk(t),         put /  'dualvar ' RK.tn(t)     ' ' marketrk.tn(t));
loop((tl,tp),           put /  'dualequ ' tkdef.tn(tl) ' ' K.tn(tp));
putclose;

*       Solve the counterfactual:
```

```
ramsey.iterlim = 10000;
SOLVE ramsey USING emp minimizing OBJ;

variable        P(t)    Output price,
                PK(t)   Capital price,
                PL(t)   Wage rate;

P.L(t) = markety.m(t);
PK.L(t)= marketk.m(t);
PL.L(t)= marketl.m(t);
RK.l(t)= marketrk.m(t);

I.L(t) = I.L(t)/i0;
K.l(t) = K.L(t)/k0;

$include report
```

A.3: The Dual EMP approach for Ramsey Model

```
$TITLE Ramsey Model -- EMP dual formulation calibrated to base year data

$include dataCES

Variables
        OBJ     Objective function;

Nonnegative Variables

* Dual variables
        RK(tt)  Rental rate
        PL      Labor price
        P       Output price
        PK(tt)  Capital price,
        EX      Expenditures

* Primal variables
        I(tt)   Investment
        K(tt)   Capital stock;

Equations

* Zero profit condition
         profity(t)         production,
         profiti(tt)        investments
         profitk            capital

* Functions
         expend             expenditures function
         objdef             objective function

* Auxiliary constraint
         tkdef              post-terminal capital stock;

objdef..             OBJ=e= sum(t,  PL(t)*qref(t)*l0)
                          + sum(tf, PK(tf)*k0)
                          - sum(tp, PK(tp)*K(tp))
                          + sum(t,  taxk(t)*RK(t)*rk0*K(t))
                          - EX;

expend..                ra0 * sum(t, thetac(t) * (P(t)/pref(t))**(1-sigma))**(1/(1-sigma)) =e= EX;
profity(t)..            PL(t)**(1-kvs) * (RK(t)*(1+taxk(t)))**kvs =e= P(t);
profitk(t(tt))..        PK(tt) =e= PK(tt+1)*(1-delta) + RK(tt)*rk0;
profiti(t(tt))..        P(tt)  =e= PK(tt+1);
tkdef(tl(t))..          I(t)/I(t-1) =e= 1+g;

model ramseyd /profity, profiti, profitk, objdef, expend, tkdef /;

OBJ.L   = 0;
I.L(t)  = i0*qref(t);
K.L(tt) = k0*qref(tt);
```

```
RK.L(t) = pref(t);
PK.L(tt)= pref(tt)*(1+r);
P.L(t)  = pref(t);
PL.L(t) = pref(t);
EX.FX   = ra0;

*        Assign the primal multipliers consistent with this growth path:
expend.m = 1;
profity.m(t)  = y0*qref(t);
profitk.m(tt) = k0*qref(tt);
profiti.m(t)  = i0*qref(t);

*        Translate NLP into MCP
file dempinfo / '%emp.info%' /;
put dempinfo;
loop((tl,tp),   put /  'dualequ ' tkdef.tn(tl)  ' ' K.tn(tp));
loop(t$taxk(t), put /  'dualvar ' K.tn(t)   ' ' profitk.tn(t));
loop(tl(t),     put /  'dualvar ' I.tn(t)   ' ' profiti.tn(t)
                    /  'dualvar ' I.tn(t-1) ' ' profiti.tn(t-1));
putclose;

$onecho >emp.opt
filename=emp-d-scalar.gms
Dict=emp-d-scalar.txt
$offecho
ramseyd.optfile = 1;

*       Replicate the benchmark equilibrium:

ramseyd.iterlim = 0;
SOLVE ramseyd USING emp minimizing OBJ;

* Apply a tax on capital inputs of 25% beginning in year 6:

taxk(t)$(t.val > 2012) = 0.25;

*       Regenerate the information file which changes when taxk increases from zero:

put dempinfo;
loop((tl,tp),   put /  'dualequ ' tkdef.tn(tl)  ' ' K.tn(tp));
loop(t$taxk(t), put /  'dualvar ' K.tn(t)   ' ' profitk.tn(t));
loop(tl(t),     put /  'dualvar ' I.tn(t)   ' ' profiti.tn(t)
                    /  'dualvar ' I.tn(t-1) ' ' profiti.tn(t-1));
putclose;

*       Solve the model:

ramseyd.iterlim = 1000;
SOLVE ramseyd USING emp minimizing OBJ;

variable        C(t)    Consumption;

*       Extract solution values for capital stock, investment and output:
```

```
I.L(t) = profiti.m(t)/i0;
K.l(t) = profitk.m(t)/k0;
C.L(t) = profity.m(t) - profiti.m(t);

$include report
```

A.4: Negishi Model with sales tax

The Primal NLP approach

```
$title A Simple Negishi model using primal NLP optimization
$include data_negishi.gms

Variables       U               utility,
                OBJ             objective;

nonnegative variables
                C(g,h)          Household consumption,
                X(f,g)          Factor inputs,
                Y(g)            Sectoral supply,
                P(g)            Market price estimate,
                THETA(h)        Negishi coefficients (welfare weights);

Equation        objdef          objective function,
                udef            utility function,
                market          good market clearing condition,
                supply          production function,
                fmarket         factor market clearing condition;


objdef..        OBJ =e= -sum(h,u0(h))*sum{h, THETA(h) * log(U(h))} + sum(g, t(g)*P(g)*Y(g));

udef(h)..       sum(g, alpha(g,h) * (C(g,h)/c0(g,h))**rho)**(1/rho) =E= U(h);

market(g)..     Y(g) =e= sum(h, C(g,h));

supply(g)..     phi(g) * prod(f, X(f,g)**beta(f,g)) =e= Y(g);

fmarket(f)..    sum(h, e(h,f)) =e= sum(g, X(f,g))  ;

*        No income balance should be declared because it will include dual variable W(f),
*        but for Primal approach we do not define dual variables (except P(g) that is fixed)

model negishi_nlp  /objdef, udef, supply, fmarket, market/;

$ondotl

C.lO(g,h)  =0.001;
X.lO(f,g)  =0.001;
U.lO(h)    =0.001;

Y.L(g)    =y0(g);
C.l(g,h)  =c0(g,h);
X.l(f,g)  =x0(f,g);

U.l(h) = 1;

*        P(g) should be fixed because it defines 'market' equation
```

```
P.FX(g) = 1;

*        THETA should be fixed because it defines 'fmarket' equation
THETA.FX(h)=sum(f, e(h,f))/sum(hh,sum(f, e(hh,f)));

t(g) = 0;


*       Load the solver into memory to be able to iterate more
*       rapidly at the expense of occupying a bit of memory:
negishi_nlp.solvelink = 2;

*execute_loadpoint 'MGEMDL_p.gdx';
*negishi_nlp.iterlim = 0;
Solve negishi_nlp minimizing OBJ using nlp;


t("brd") = 0.5;
*        To make a simulation for NLP we need to create a loop, because THETA.FX and P.FX
nonnegative variables
       W(f)            factor price
       INCOME          equilibrium income  ;

set             iter    Negishi iterations /iter0*iter20/;
parameter       itrlog  Iteration log;

*       Randome starting point:
INCOME.FX(h)= uniform(0,100);  ;

loop(iter,
        P.FX(g)  =  market.m(g);
        THETA.FX(h) = INCOME.L(h)/sum(h.local,INCOME.L(h));
        itrlog(iter,h) = THETA(h) * 100;
        itrlog(iter,g) = Y(g);
        solve negishi_nlp using nlp minimizing OBJ;

        W.FX(f)  =  fmarket.m(f);
        INCOME.FX(h)= sum(f, W(f)*e(h,f)) + (1/card(h)) * sum(g, P(g)*Y(g)*t(g)););
display itrlog, W.L, INCOME.L;

$setglobal domain iter
set iterlbl(iter) /iter0 0, iter5 5, iter10 10, iter15 15, iter20 20/;
$setglobal labels iterlbl
$libinclude plot itrlog
```

The Primal EMP approach

```
$title A Simple Negishi model using primal EMP optimization
$include data_negishi.gms
```

```
Variables       U               utility,
                OBJ             objective;

nonnegative variables
                C(g,h)          Household consumption,
                X(f,g)          Factor inputs,
                Y(g)            Sectoral supply,

*       Dual variables introduced through side constraints:
                P(g)            Market price estimate,
                W(f)            factor price
                INCOME(h)       Equilibrium income
                THETA(h)        Negishi coefficients;

Equation        objdef          objective function,
                udef            utility function,
                market          good market clearing condition,
                supply          production function,
                fmarket         factor market clearing condition,
                incomedef       Defines income,
                thetadef        Defines income share;

objdef..        OBJ =e= -sum(h,u0(h))*sum{h, THETA(h) * log(U(h))} + sum(g, t(g)*P(g)*Y(g));

*       Primal constraints which would appear in the NLP model:

udef(h)..       sum(g, alpha(g,h) * (C(g,h)/c0(g,h))**rho)**(1/rho) =E= U(h);

market(g)..     Y(g) =e= sum(h, C(g,h));

supply(g)..     phi(g) * prod(f, X(f,g)**beta(f,g)) =e= Y(g);

fmarket(f)..    sum(h, e(h,f)) =e= sum(g, X(f,g));

*       Side constraints for the EMP model (we do not need it for NLP):

incomedef(h)..  INCOME(h) =e= sum(f, W(f)*e(h,f)) + (1/card(h)) * sum(g, P(g)*Y(g)*t(g));

thetadef(h)..   THETA(h)*sum(hh,INCOME(hh)) =e= INCOME(h);

model negishi_emp  /objdef, udef, supply, fmarket, market, incomedef, thetadef/;

file olga / '%emp.info%' /;

C.lO(g,h)  =0.001;
X.lO(f,g)  =0.001;
U.lO(h)    =0.001;

Y.L(g) = y0(g);
C.l(g,h)  =c0(g,h);
X.l(f,g)  =x0(f,g);
P.L(g) = 1;
W.L(f) = 1;
INCOME.L(h) = sum(f,e(h,f));
```

```
* Distribute Negishi coefficients uniformly (starting point)
THETA.L(h) = 1/card(h);
U.l(h)    = 1;



*       Replicate the benchmark equilibrium -- all prices should be unity:
t(g) = 0;

put olga ;
loop{f,      put /  'dualvar ' W.tn(f)        ' ' fmarket.tn(f)};
loop{g$t(g), put /  'dualvar ' P.tn(g)        ' ' market.tn(g)};
loop{h,      put /  'dualequ ' thetadef.tn(h) ' ' THETA.tn(h)};
loop{h,      put /  'dualequ ' incomedef.tn(h)' ' INCOME.tn(h)};
putclose;
Solve negishi_emp minimizing OBJ using emp;


*       Solve a counter-factual in which we apply a 50% excise tax:
t("brd") = 0.5;

put olga ;
loop{f,      put /  'dualvar ' W.tn(f)        ' ' fmarket.tn(f)};
loop{g$t(g), put /  'dualvar ' P.tn(g)        ' ' market.tn(g)};
loop{h,      put /  'dualequ ' thetadef.tn(h) ' ' THETA.tn(h)};
loop{h,      put /  'dualequ ' incomedef.tn(h)' ' INCOME.tn(h)};
putclose;
Solve negishi_emp minimizing OBJ using emp;
```

Faculty of Economic Sciences
University of Warsaw
44/50 Długa St.
00-241 Warsaw
www.wne.uw.edu.pl